PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #42

April 3, 1982

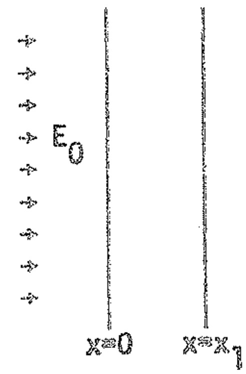
General Instructions

This Comprehensive Examination for Spring 1982 (#42) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit---especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

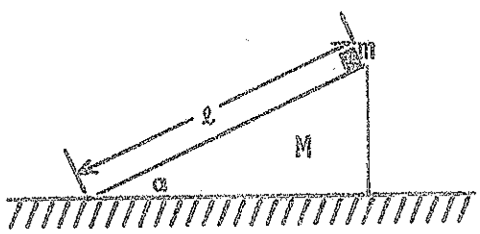
1. The dielectric constant $K(x)$ of a semi-infinite uncharged slab varies linearly between the values $K = 1$ at $x = 0$ and $K = 2$ at $x = x_1$, where it suddenly drops to $K = 1$ again for all $x > x_1$. A uniform electric field $E_0\hat{i}$ is then applied perpendicularly to the left-hand surface of the slab.



(10) (a) Determine the electric field for $x > 0$.

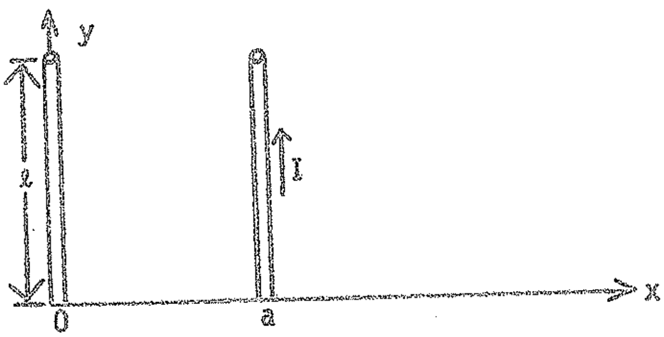
(10) (b) Determine the polarization charge density everywhere.

(20) 2.



A block of mass m is placed at the top of a wedge whose mass is M and whose diagonal dimension is ℓ. There is no friction between m and M, or between M and the horizontal surface. Regarding m as a point, determine the time it takes to slide to the bottom of the wedge, starting from rest.

3. A copper wire of small cross-section coincides with the y-axis from $y = 0$ to $y = \ell$. A similar wire is placed parallel to the first at $x = a$.



(8) (a) Calculate the vector potential $\vec{A}$ at the point $(0,y)$ where $0 < y < \ell$ in the first wire, due to a steady current $\vec{I} = I\hat{k}$ in the second wire.

(8) (b) Show in detail how you would calculate the mutual inductance M between these wires, if you were <u>given</u> $\vec{A}(y)$. (You may assume that the circuit carrying I closes at infinity from leads in the z-direction.)

(4) (c) Calculate M for the case (a).

Hints: The equation obeyed by the electrostatic potential ϕ in free space

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}\text{, has solution } \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|} .$$

The vector potential $\vec{A}$ for a steady current and no permeable materials obeys $\nabla^2\vec{A} = -\mu_0\vec{J}$. (Be careful; this is a <u>vector</u> equation.)

You may find one or both of the following integrals useful:

$$\int \sinh^{-1}\theta\, d\theta = \theta \sinh^{-1}\theta - \sqrt{1-\theta^2} \qquad \int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$

4. Consider an imperfect gas of $N \to \infty$ particles whose partition function is

$$Q_N(V,T) = \int \frac{d^{3N}p \, d^{3N}q}{N! \, h^{3N}} e^{-\beta H(p,q)}$$

and which is contained in a volume $V \to \infty$. The system's Hamiltonian is given by

$$H = \sum_{i=1}^{3N} \frac{p_i^2}{2m} + \sum_{i<j} \phi_{ij} (|\vec{r}_i - \vec{r}_j|).$$

(14) (a) By defining $f_{ij}(r) = e^{-\beta\phi(r)} - 1$ where $r = |\vec{r}_i - \vec{r}_j|$

show that the equation of state of the gas is given by

$$\frac{Pv}{kT} = 1 + v\frac{\partial}{\partial v} z(v,T)$$

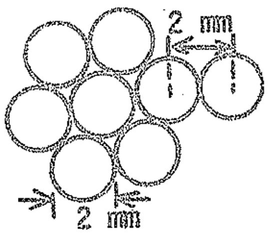
where $v = \frac{V}{N}$ and $z = \frac{1}{N} \ln [\frac{1}{V^N} \int d^3r_1 \ldots . d^3r_N \prod_{i<j} (1 + f_{ij})]$.

Note that z is <u>not</u> the partition function, and that

$$\int_0^\infty e^{-\alpha^2 x^2} dx = \frac{\sqrt{\pi}}{2\alpha} .$$

(6) (b) Also, knowing that the Jacobian, J, associated with the transformation from individual particle coordinates r_i and r_j to center of mass and relative separation coordinates R and r has the value of unity (i.e., $d^3r_i \, d^3r_j = J \, d^3R \, d^3r$), determine an approximation for $z(v,T)$ that neglects powers of $f(r)$ greater than the first.

(10) 5. The impressionist painter Georges Seurat was a pointillist; his paintings consist of an enormous number of closely spaced small circular dots of pure pigment about 2 mm in diameter. The illusion of color mixing occurs only at the observer's eye. Discuss this process, and calculate how far from such a painting one should stand in order to perceive the desired color blending.



6. Consider an ion having total spin $S = 1$ to be situated in a crystal which has axial symmetry, such that the Hamiltonian is

$$H = -g_{\parallel} \mu_0 B_z S_z - g_{\perp} \mu_0 (B_x S_x + B_y S_y) \qquad (\mu_0 = \text{Bohr magneton})$$

when the ion is subjected to a uniform external magnetic induction field $\vec{B} = B_0 \hat{k}$ in the z-direction. The eigenstates are

$$|M_S = 1\rangle \qquad |M_S = 0\rangle \qquad \text{and} \qquad |M_S = -1\rangle$$

with corresponding eigen-energies

$$g_{\parallel} \mu_0 B_0 , \qquad 0 \qquad , \qquad -g_{\parallel} \mu_0 B_0 .$$

Now suppose the field to be re-oriented, so that $\vec{B} = \dfrac{B_0}{\sqrt{2}} (\hat{i} + \hat{j})$ has equal components in the x and y directions only. Given that the spin raising and lowering operators are defined by

$$S_{\pm}|M_S\rangle = [S(S+1) - M_S(M_S \pm 1)]^{1/2} \; |M_S \pm 1\rangle$$

subject to the usual restrictions $\quad S_+|M_S = S\rangle = S_-|M_S = -S\rangle = 0$

$$\text{and} \qquad S_z|M_S\rangle = M_S|M_S\rangle .$$

(10) (a) Determine the energies allowed to this system.

(-) (b) Determine the normalized eigenstates of this system.

(5) (c) Show that the eigenstates are orthogonal to each other.

(1) *Solution*

(a) $\nabla\cdot D = \nabla\cdot(KE) = \frac{\partial K}{\partial X}E + K\frac{\partial E}{\partial X} = 0$

$$K = 1 + \frac{X}{X_0} \qquad 0<X<X_0$$

$$= 1 \qquad X > X_0$$

$$\frac{\partial K}{\partial X} = \frac{1}{X_0}$$

$$\frac{1}{X_0}E + \left(1+\frac{X}{X_0}\right)\frac{dE}{dX} = 0$$

$$\frac{dE}{E} = -\frac{dX}{X_0\left(1+\frac{X}{X_0}\right)} = -\frac{dX}{X+X_0}$$

$$\ln E = -\ln(X+X_0) + \ln C$$

$$E = \frac{C}{X+X_0} = \frac{E_0 X_0}{X+X_0} \qquad 0<X<X_0$$

at X_0, $E_{in} = \frac{E_0}{2}$ $\qquad E_{out} \qquad K$

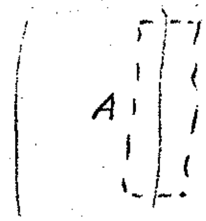
$$2E_{in} = 1\,E_{out} \qquad E_{out} = E_0$$

(b) $D = E + 4\pi P = E_0$ by continuity

$$P = \frac{D-E}{4\pi} = \frac{E_0}{4\pi}\left(1 - \frac{X_0}{X+X_0}\right) = \frac{E_0 X}{4\pi(X+X_0)}$$

$$\rho_p = -\nabla\cdot P = -\frac{E_0}{4\pi}\,\frac{X+X_0-X}{(X+X_0)^2} = \frac{-1}{4\pi}\frac{E_0 X_0}{(X+X_0)^2} \qquad 0<X<X_0$$

at $X \lesssim X_0$ $\quad P = \dfrac{E_0}{8\pi}$, at $X \gtrsim X_0$ $\quad P = 0$



$$\int \nabla \cdot P \, d\tau = -q$$

$$-\frac{E_0}{8\pi} A = -q$$

$$\frac{q}{A} = \frac{E_0}{8\pi}$$ on the right surface

alternative b solution

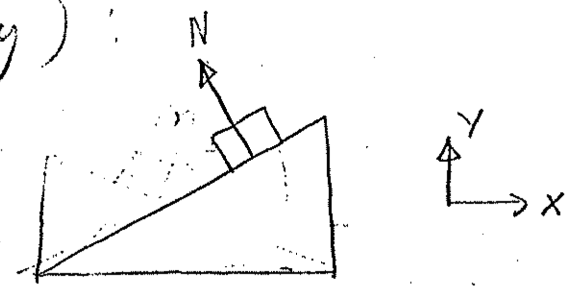
$$4\pi \rho = \nabla \cdot E = -\frac{E_0 X_0}{(X + X_0)^2}$$

at right surface $\int E \cdot ds = A\left(E_0 - \dfrac{E_0}{2}\right) = 4\pi q$

$$\sigma = \frac{q}{A} = \frac{1}{8\pi} E_0$$

(2)

Solution 1 (Elementary):



Let (1) a' be the magnitude of the accel of m w.r.t. M

(2) $A \equiv$ accel of M

then $A - a'\cos\alpha = a_X$

$-a'\sin\alpha = a_Y$

(3) $N =$ normal force on m $\quad N_X = -N\sin\alpha$, $\quad N_Y = +N\cos\alpha$

$-N =$ " " " M

$N_X = m a_X$ (a) $-N\sin\alpha = m(A - a'\cos\alpha)$

$N_Y - mg = m a_Y$ (b) $N\cos\alpha - mg = -ma'\sin\alpha$

$-N_X = MA$ (c) $N\sin\alpha = MA$

$$-N\sin\alpha = m\left(\frac{N\sin\alpha}{M} - a'\cos\alpha\right)$$

$$\ominus N\sin\alpha\left(1 + \frac{m}{M}\right) = \ominus m a'\cos\alpha$$

$$N\cos\alpha = mg - ma'\sin\alpha$$

$$\gamma \equiv \tan\alpha\left(1 + \frac{m}{M}\right) = \frac{m a'\cos\alpha}{mg - m a'\sin\alpha}$$

$$\gamma\left[g - a'\sin\alpha\right] = a'\cos\alpha$$

$$a'(\gamma\sin\alpha + \cos\alpha) = \gamma g$$

$$a' = \frac{\gamma g}{\gamma\sin\alpha + \cos\alpha} = \frac{g}{\sin\alpha + \frac{\cos\alpha}{\gamma}}$$

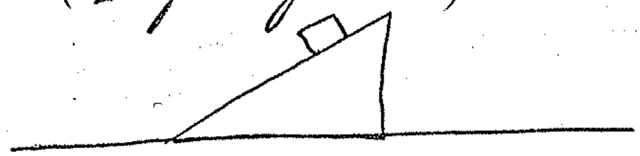
$$= \frac{g}{\sin\alpha + \frac{\cos^2\alpha}{\sin\alpha}\frac{M}{M+m}} = \frac{g\sin\alpha}{\sin^2\alpha + \frac{M}{M+m}\cos^2\alpha}$$

$$= \frac{g \sin \alpha}{1 - \cos^2\alpha \left(1 - \frac{M}{M+m}\right)} = \frac{g \sin \alpha}{1 - \cos^2\alpha \, \frac{m}{M+m}}$$

$$l = \frac{1}{2} a' t^2$$

$$t = \sqrt{\frac{2l}{a}} = \sqrt{\frac{2l \left(1 - \cos^2\alpha \, \frac{m}{M+m}\right)}{g \sin \alpha}}$$

Solution 2 (Lagrangian)



$$L = \frac{1}{2} m\left[(V_x + v_x)^2 + v_y^2\right] - mgY$$

$$Y = s \sin\alpha$$

$$v_Y = v \sin\alpha$$

$$v_x = v\cos\alpha \qquad \underline{\underline{V_x = V}}$$

$$L = \frac{1}{2} M V^2 + \frac{1}{2} m\left[(v\cos\alpha + V)^2 + v^2 \sin^2\alpha\right] - mg\, s\sin\alpha$$

$$= \quad '' \quad + \frac{1}{2} m\left[v^2 + 2vV\cos\alpha + V^2\right] - mg\, s\sin\alpha$$

$$P = \frac{\partial L}{\partial V} = (m+M)V + mv\cos\alpha = \text{const since } X \text{ is cyclic} = 0$$

$$\frac{\partial L}{\partial v} = mv + mV\cos\alpha \quad , \quad \frac{\partial L}{\partial s} = -mg\sin$$

$$m(\dot{v} + \dot{V}\cos\alpha) + mg\sin\alpha = 0$$

$$\cancel{m}\left(\dot{v} - \cos\alpha \frac{m}{M+m} \dot{v}\cos\alpha\right) + \cancel{m}g\sin\alpha = 0$$

$$v = \frac{-g\sin\alpha\, t}{1 - \frac{m}{M+m}\cos^2\alpha} + \overset{=0}{v_0} \qquad \dot{v} = \frac{g\sin\alpha}{1 - \frac{m}{M+m}\cos^2\alpha}$$

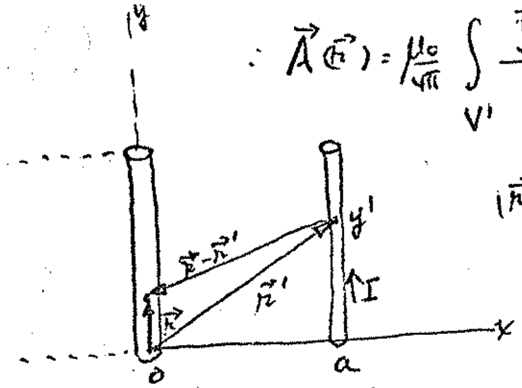
$$l = \frac{1}{2} a t^2 \qquad t = \sqrt{\frac{2l}{a}} = \sqrt{\frac{2l\left(1 - \frac{m}{M+m}\cos^2\alpha\right)}{g\sin\alpha}}$$

Solution: (3) Just as $\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|}$, Each component of $\vec{A}$, say A_x, will be

$A_x(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{J_x(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|}$ and these solutions can be added together to form $\vec{A}$:

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|}$$

(μ_0 appears because copper is non-magnetic)



$$|\vec{r}-\vec{r}'| = \sqrt{(y'-y)^2+a^2}$$

If A is the (small) cross-section of the wire, $\vec{J} = \frac{I}{A}\hat{\jmath}$

and $\vec{A} = \frac{\mu_0}{4\pi}\left(\frac{I}{A}\right)\int_{V'}\frac{dv'}{|\vec{r}-\vec{r}'|}\;\hat{\jmath}$

The denominator is independent of x' and z' for small A, so integration of $dv' = dx'dy'dz'$ with respect to x' and z' give merely $\int dx'\int dy' = A$

(This is an acceptable starting point).

a) and so $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I}{\cancel{A}}\cancel{A}\int_0^l \frac{dy}{\sqrt{(y'-y)^2+a^2}}\;\hat{\jmath}$

Since $\int\frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$, $\vec{A} = \frac{\mu_0 I}{4\pi}\left[\sinh^{-1}\left(\frac{l-y}{a}\right) - \sinh^{-1}\left(\frac{-y}{a}\right)\right]$

$$\text{or } \vec{A}(y) = \frac{\mu_0 I}{4\pi}\left[\sinh^{-1}\left(\frac{y}{a}\right) - \sinh^{-1}\left(\frac{y}{a}-\frac{l}{a}\right)\right]$$

b) Since $\vec{B} = \nabla\times\vec{A}$, $\Phi \equiv \int_S \vec{B}\cdot\hat{n}\,da = \int_S \nabla\times\vec{A}\cdot\hat{n}\,da = \oint_C \vec{A}\cdot d\vec{l}$ from Stokes' Theorem, where C is any closed contour which bound

If we choose, say, the dashed contour, which closes at infinity, the only non-vanishing contribution to the flux, since $\vec{A} = A_y\hat{\jmath}$ only, is $\Phi = \int_0^l A(y)\,dy$

$$\text{then } M_{12} \equiv \frac{\partial\Phi}{\partial I_2} = \frac{1}{I}\int_0^l A(y)\,dy$$

c) In this case, $M_{12} = \frac{1}{I}\int_0^l A(y)\,dy = \frac{\mu_0}{4\pi}\left[\int_0^l \sinh^{-1}\left(\frac{y}{a}\right)dy - \sinh^{-1}\left(\frac{y}{a}-\frac{l}{a}\right)dy\right]$

$$= \frac{\mu_0 a}{4\pi}\left[\int_0^{l/a}\sinh^{-1}\left(\frac{y}{a}\right)d\left(\frac{y}{a}\right) - \sinh^{-1}\left(\frac{y}{a}-\frac{l}{a}\right)d\left(\frac{y}{a}\right)\right]$$

$$= \frac{\mu_0 a}{4\pi}\left[\frac{l}{a}\sinh^{-1}\left(\frac{l}{a}\right) - \sqrt{1\cancel{+\left(\frac{l}{a}\right)^2}} - 0 + \sqrt{1\cancel{+\left(\frac{l}{a}\right)^2}}\right]$$

so $M_{12} = \frac{\mu_0 l}{4\pi}\sinh^{-1}\frac{l}{a}$

(4)

Solution:

(a) $$Q_N(V,T) \equiv \int \frac{d^{3N}p\, d^{3N}q}{N!\, h^{3N}}\, e^{-\beta \mathcal{H}(p,q)}$$

and $$e^{-\beta F(V,T)} = Q_N(V,T),$$ where F is the free energy.

$$\therefore \quad F(V,T) = -\frac{1}{\beta} \ln Q_N(V,T)$$

and $$p = -\left(\frac{\partial F}{\partial V}\right)_T \quad \therefore \quad \frac{p}{kT} = \frac{\partial}{\partial V} \ln Q_N(V,T)\Big|_T$$

or $$\frac{p}{kT} = \frac{1}{N}\cdot\frac{\partial}{\partial v} \ln Q_N(V,T)\Big|_T \qquad \text{since } v = \frac{V}{N}, \quad \frac{\partial v}{\partial V} = \frac{1}{N}.$$

Our problem requires:

$$Q_N(V,T) = \frac{1}{N!\, h^{3N}} \int d^{3N}p\, d^{3N}r\, e^{-\beta\left(\sum_{i=1}^{3N} p_i^2/2m + \sum_{i<j} \phi_{ij}\right)}$$

$$= \frac{1}{N!\, h^{3N}} \left[\int_{-\infty}^{\infty} d^{3N}p_j\, e^{-\beta \sum_{j=1}^{3N} p_j^2/2m} \int d^{3N}r\, e^{-\beta \sum_{i<j} \phi_{ij}} \right]$$

$$= \frac{1}{N!\, h^{3N}} \left[\int_{-\infty}^{\infty} dp_j\, e^{-\beta p_j^2/2m} \right]^{3N} \left[\int d^3r_1 \ldots d^3r_N\, e^{-\beta \sum_{i<j} \phi_{ij}} \right]$$

$$= \frac{1}{N!\, h^{3N}} \left[\frac{2\pi m}{\beta} \right]^{3N} \left[\int d^3r_1 \ldots d^3r_N\, e^{-\beta \sum_{i<j} \phi_{ij}} \right]$$

in terms of f_{ij} $\rightarrow$ $e^{-\beta \sum_{i<j} \phi_{ij}} = \prod_{i<j} (1+f_{ij})$

$$\therefore \quad Q_N(V,T) = \frac{1}{N!\, h^{3N}} \left(\frac{2\pi m}{\beta} \right)^{3/2} \underbrace{\left[\int d^3 r_1 \cdots d^3 r_N \prod_{i<j} (1+f_{ij}) \right]}_{X}$$

$$\therefore \frac{p}{kT} = \frac{1}{N} \frac{\partial}{\partial v} \ln Q_N(V,T) \Big|_T = \frac{1}{N} \frac{\partial}{\partial v} \ln X \Big|_T$$

given $Z(v,T) = \dfrac{1}{N} \ln \left[\dfrac{1}{V^N} \displaystyle\int d^3 r_1 \cdots d^3 r_N \prod_{i<j} (1+f_{ij}) \right]$

$$= \frac{1}{N} \ln \left[\frac{1}{V^N} \cdot X \right]$$

$$N\, Z(v,T) = \ln X - N \ln V$$

or $\quad \ln X = N\, Z(v,T) + N \ln V$

and $\quad \dfrac{p}{kT} = \dfrac{1}{N} \cdot \dfrac{\partial}{\partial v} \left[N\, Z(v,T) + N \ln V \right]$

$$\therefore \quad \frac{p}{kT} = \frac{1}{V} \frac{\partial V}{\partial v} + \frac{\partial Z}{\partial v}(v,T) = \frac{1}{v} + \frac{\partial Z}{\partial v}(v,T)$$

and $\quad \dfrac{pv}{kT} = 1 + v \dfrac{\partial}{\partial v} Z(v,T)$

(b) $$Z(v,T) = \frac{1}{N} \ln\left[\frac{1}{V^N}\int d^3r_1 \cdots d^3r_N \prod_{i<j}(1+f_{ij})\right]$$

$$Z(v,T) \cong \frac{1}{N} \ln\left[1 + \frac{1}{V^N}\,\frac{N(N-1)}{2}\,V^{N-2}\int d^3R\, d^3r\, f(r)\right]$$

$$\cong \ln\left[1 + \frac{1}{2}\,\frac{N(N-1)}{V}\int d^3r\, f(r)\right]^{1/N}$$

$$Z(v,T) \cong \ln\left[1 + \frac{1}{2}\,\frac{N}{v}\int d^3r\, f(r)\right]^{1/N}.$$

(5) a) When the painting is illuminated with white light, each circular dot becomes an extended source. Because the spectral purity of white light is so poor, $\frac{\Delta\nu}{\bar{\nu}} = \frac{700-400}{550} = \frac{\Delta\lambda}{\bar{\lambda}} \doteq 0.5$, the coherence length is extremely small:

$$\Delta\nu = \bar{\nu}\left(\frac{\Delta\lambda}{\bar{\lambda}}\right) = \frac{c}{\bar{\lambda}}\left(\frac{\Delta\lambda}{\bar{\lambda}}\right) \text{ and since } \Delta\nu\,\Delta t \sim 1,$$

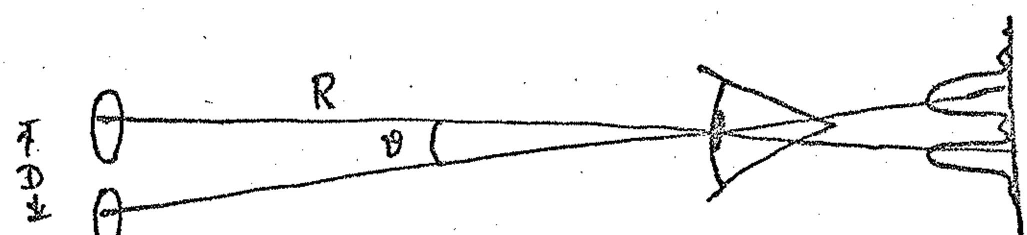
the coherence length is roughly $\Delta x = c\,\Delta t = \frac{c}{\Delta\nu} = \frac{c}{c}\frac{\bar{\lambda}}{\left(\frac{\Delta\lambda}{\bar{\lambda}}\right)} \sim \frac{5.5\times10^{-7}\,m}{0.5} \sim \underline{10^{-6}\,m}$

So even parts of a single dot separated by this distance are incoherent.

b) "Reflected" light from each dot travels to the viewer's retina; there an image is formed by the eye's lens. The image of each point of the dot is really a diffraction pattern resulting from the limited aperture of the pupil ($d \approx 2$ mm). Most of the irradiance in the pattern is in the central ring of this Airy pattern.

This central maximum has angular width $\underline{\alpha = 1.22\,\frac{\lambda}{d}}$

where λ = wavelength of light $\approx 5.5\times10^{-7}$ m, and d is the limiting aperture diameter.



c) There will be substantial overlap of neighboring dot-images (and therefore substantial color-mixing) when the Airy maximum of a point on one dot falls, say, at twice the angle of the first Airy minimum of the corresponding point on a neighboring dot. (The Rayleigh Criterion would give almost perfect overlap).

d) In the small angle approx., if D is the separation between corresponding points of neighboring dots, we require

$$\vartheta = \frac{D}{R} = 2\alpha = 2\left(1.22\,\frac{\lambda}{d}\right)$$

$$\text{so } R = \frac{dD}{2.44\,\lambda} = \frac{(2\times10^{-3}\,m)(2\times10^{-3}\,m)}{2.44\,(5.5\times10^{-7}\,m)} \approx \underline{3 \text{ meters}}$$

(any calculation that gives $2\,m < R < 4\,m$ is OK)

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(i) $\mathcal{H} = \mu_0 \tilde{g} \cdot H \cdot \tilde{S} = \mu_0 \left(g_{\parallel} H_z S_z + g_{\perp} [H_x S_x + H_y S_y] \right)$

$\therefore \quad \mathcal{H} = g_{\perp} \mu_0 \dfrac{H_0}{\sqrt{2}} (S_x + S_y)$ for our case.

But $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_+ - S_-)$

$$\therefore \quad \mathcal{H} = \frac{g_{\perp} \mu_0 H_0}{\sqrt{2}} \left[\left(\frac{1-i}{2}\right) S_+ + \left(\frac{1+i}{2}\right) S_- \right]$$

Let's call $g_{\perp} \mu_0 H_0 / \sqrt{2} = C$.

Then

$$\begin{array}{cccc}
 & |1\rangle & |0\rangle & |-1\rangle \\
\langle 1| & 0 & C(2-0)^{1/2}\left(\frac{1-i}{2}\right) & 0 \\
\langle 0| & C\left(\frac{1+i}{2}\right)(2-0)^{1/2} & 0 & C\left(\frac{1-i}{2}\right)(2-0)^{1/2} \\
\langle -1| & 0 & C\left(\frac{1+i}{2}\right)(2-0)^{1/2} & 0
\end{array}$$

i.e.

$$\begin{pmatrix}
0 & g_{\perp}\mu_0 H_0 \left(\frac{1-i}{2}\right) & 0 \\
g_{\perp}\mu_0 H_0 \left(\frac{1+i}{2}\right) & 0 & g_{\perp}\mu_0 H_0 \left(\frac{1-i}{2}\right) \\
0 & g_{\perp}\mu_0 H_0 \left(\frac{1+i}{2}\right) & 0
\end{pmatrix}$$

To get eigen values of energy, we must diagonalize this matrix

$$\begin{vmatrix} -\lambda & g_\perp \mu_0 H_0 \left(\frac{1-i}{2}\right) & 0 \\ g_\perp \mu_0 H_0 \left(\frac{1+i}{2}\right) & -\lambda & g_\perp \mu_0 H_0 \left(\frac{1-i}{2}\right) \\ 0 & g_\perp \mu_0 H_0 \left(\frac{1+i}{2}\right) & -\lambda \end{vmatrix} = 0$$

which yields,

$$\lambda^3 - g_\perp^2 \mu_0^2 H_0^2 \lambda = 0 \quad \text{and} \quad \therefore \lambda = 0 \;, \; \lambda = \pm g_\perp \mu_0 H_0 .$$

as energy eigenvalues.

(ii) Returning to our eigenvalue statement, we have for $\underline{\lambda = +g_\perp \mu_0 H_0}$

$$0\, a_1 + g_\perp \mu_0 H_0 \left(\frac{1-i}{2}\right) a_0 + 0\, a_{-1} = g_\perp \mu_0 H_0\, a_1$$

$$g_\perp \mu_0 H_0 \left(\frac{1+i}{2}\right) a_1 + 0\, a_0 + g_\perp \mu_0 H_0 \left(\frac{1-i}{2}\right) a_{-1} = g_\perp \mu_0 H_0\, a_0$$

$$0\, a_1 + g_\perp \mu_0 H_0 \left(\frac{1+i}{2}\right) a_0 + 0\, a_{-1} = g_\perp \mu_0 H_0\, a_{-1}$$

where a_1, a_0 and a_{-1} are the coeffs. of $|1\rangle$, $|0\rangle$ and $|-1\rangle$ needed to make up the new eigenstates.

From these equations we get $a_{-1} = i a_1$
and $a_0 = (1+i) a_1$

$\therefore \; |1'\rangle = a_1|1\rangle + a_0|0\rangle + a_{-1}|-1\rangle$
with $|a_1|^2 + |a_0|^2 + |a_{-1}|^2 = 1.$ (NORMALIZATION)

and so $a_1^2 + (1+i)(1-i)a_1^2 + i(-i)a_1^2 = 1$
and $\therefore \; a_1 = \frac{1}{2}$

$$\therefore |1'\rangle = \frac{1}{2}|1\rangle + \left(\frac{1+i}{2}\right)|0\rangle + \frac{i}{2}|-1\rangle$$

Redoing eigenstatement for $\lambda = 0$ yields

$a_0 = 0$, $a_{-1} = -i a_1$ and so

$$|0'\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}|-1\rangle$$

Also for $\lambda = -g_\perp \mu_0 H_0$ we get $a_0 = -(1+i)a_1$
and $a_{-1} = i a_1$

and $\therefore \; |-1'\rangle = \frac{1}{2}|1\rangle - \left(\frac{1+i}{2}\right)|0\rangle + \frac{i}{2}|-1\rangle$

(iii) Orthogonality:

$$\langle -1' | 1' \rangle = \frac{1}{4} - \frac{(1-i)(1+i)}{4} - \frac{i}{2}\frac{i}{2} = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

$$\langle -1' | 0' \rangle = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 0$$

$$\langle 0' | 1' \rangle = \frac{1}{2\sqrt{2}} - \left(\frac{-i}{\sqrt{2}}\right)\frac{i}{2} = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 0.$$