General Instructions

This Comprehensive Examination for Winter 1982 (#41) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.
1. A circuit with a battery with static potential difference $V_B$, internal resistance $r$, a resistor $R$, and two capacitors $C_1$ and $C_2$, initially uncharged, is switched on at $t = 0$.

![Circuit Diagram]

(a) Write down a set of equations for the unknown currents $i$, $i_1$, $i_2$, and charges $q_1$ (on $C_1$) and $q_2$ (on $C_2$).

(b) Determine the currents and charges at $t = 0$.

(c) Determine the currents and charges at $t = \infty$. 

(10) 

(5) 

(5)
2. In a one-dimensional lattice of length \( L \) the periodic potential energy can be broken into two pieces, a large constant background \( V_0 \) and a small remaining periodic part \( V \) of period \( a = L/n \) which is zero on the average:

\[
V = V_0 + \tilde{V} = V_0 + \sum_{p=0}^{N-1} v(x - pa)
\]

\[
v(x) \neq 0, \ 0 < x < a \\
v(x) = 0, \ \text{otherwise}
\]

\[
\int v(x) dx = 0
\]

(a) Determine the unperturbed eigenstates of a particle of mass \( \mu \) in this potential using periodic conditions at the lattice boundary.

(b) What energy degeneracies exist in the unperturbed system?

(c) Determine the first-order energy shift.

(d) Show that the presence of \( \tilde{V} \) gives rise to an energy gap and determine the gap condition.

(e) Sketch the energy spectrum as a function of quantum number.

**Hint:** \[ 1 + x + \ldots + x^N = \frac{1 - x^{N+1}}{1 - x} \]
3. A solid at absolute temperature $T$ is placed in an external magnetic field $B = 30,000$ gauss. The solid contains weakly interacting paramagnetic atoms of spin $1/2$ so that the energy of each atom is either $+\mu B$ or $-\mu B$. If the magnetic moment $\mu$ is equal to one Bohr magneton ($0.927 \times 10^{-20}$ erg/gauss), estimate below what temperature one must cool the solid so that more than 75% of the atoms are polarized with their spins parallel to the external magnetic field? (A rough numerical answer is required for full credit.)

4. A small hemispherical bump (of radius $a$) is raised on the inside of one conducting plate of a parallel-plate capacitor whose plates (each of infinite area $A$) are separated by a distance $d >> a$. The space between the plates is filled with a very leaky dielectric which acts like an ohmic medium of conductivity $\sigma$ and permittivity $\varepsilon$. The plate with the bump is grounded; the other is held at potential $V$. A steady current $I$ flows between the plates; the current density at the far plate is uniform and constant, as shown. Calculate the electrical force tending to pull the bump from the plate.

(HINT: Zonal Harmonics have the forms $r^n p_n(\theta)$ and $\frac{p_n(\theta)}{r^{n+1}}$.)

\[ \epsilon \]
5. A quantum mechanical particle of mass \( m \) moving in a potential \( V(\vec{r}) \) which is finite everywhere, experiences a uniform force \( \vec{F} \) which is very large, during a very short time interval, \( t = 0 \) to \( t \), such that the impulse \( \vec{A} = \vec{F} \cdot t \) is finite. The work done by this force is much greater than the mechanical energy of the unperturbed system.

(a) Given the wave function \( \psi(x, 0) \), assumed normalized, determine \( \psi(x, t) \) after the "shock". (Be careful: \( \vec{F} \) is too large to try perturbation theory.)

(b) Calculate the change in the expectation value of the particle's linear momentum \( \hat{p} \) as a result of the impulse. Comment on the result.

(c) Apply these results to the one-dimensional simple harmonic oscillator. Suppose the system is initially in its ground state; calculate the probability \( \psi_n \) that as a result of the impulse the system will be found in an eigenstate, bound, with energy \( E_n \).

(For the harmonic oscillator, \( V(x) = \frac{1}{2} m \omega^2 x^2 \), the normalized eigenfunctions are \( \psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m \omega}{\hbar} \right)^{1/4} e^{-x^2/2} H_n(x) \) where \( \xi = \frac{m \omega}{\hbar} x \).)

\( H_n \) are the Hermite Polynomials (\( H_0(\xi) = 1 \)), which obey the relation

\[
\int_{-\infty}^{\infty} e^{-\xi^2} H_n(\xi) e^{a\xi} d\xi = a^n \int_{-\infty}^{\infty} e^{-\xi^2} e^{a\xi} d\xi.
\]

You may also need \( \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \).

6. Consider a thumb tack of mass \( M \) having a shaft of length \( l \) and a head of radius \( R \) spinning on its point (assumed to be fixed). For purposes of this problem you can consider (1) the head to be a disc in shape, and (2) the shaft to be of negligible mass compared to the head. Show that the maximum precessional velocity of the thumb tack is given by \( \omega_p \max = \frac{2g(2/l)^2}{mR[1 + (l/R)^2]^{1/2}} \), where \( \omega \) is the angular velocity of the tack about an axis through its shaft.
(a) 
\[ i = i_1 + i_2 \]
\[ V_B - i_1 r - \frac{d}{dt} \frac{q_1}{C_1} = 0 \]
\[ V_B - i_1 r - i_2 R - \frac{d}{dt} \frac{q_2}{C_2} = 0 \]
\[ i_1 = \frac{dq_1}{dt}, \quad i_2 = \frac{dq_2}{dt} \]

(b) \[ q_1 = q_2 = 0 \]
\[ V_B - i_1 r = 0 \]
\[ V_B - i_1 r - i_2 R = 0 \]
\[ i_1 = \frac{V_B}{r}, \quad i_2 = 0 \]
\[ i_2 = \frac{V_B}{r} \]

(c) Capacitors will change \[ i_1 = i_2 = 0 = i \]
\[ \frac{q_1}{C_1} = \frac{q_2}{C_2} = \frac{V_B}{C_1} = \frac{V_B}{C_2} = V_B \]
\[ q_1 = C_1 V_B, \quad q_2 = C_2 V_B \]
For the upper sign $E^{(2)}$ is negative and for the lower sign it is positive. The spectrum looks like the figure. The energies fall below the unperturbed values for $n \leq \frac{jN}{2}$ and above for $n > \frac{jN}{2}$. In between there are gaps.

If the energy denominator is small compared to the potential matrix element, degenerate or near degenerate perturbation theory must be used.

Only states satisfying the condition $m = n - jN$ are connected by $V$. Thus

$$\psi = a \psi_n + b \psi_{n-jN}$$

leading to an eigenvalue condition since $V_{nn} = 0$

$$|E^{(0)}_n - \lambda |V_{mn}| = 0 \quad m = n - jN$$

$$\lambda^2 - (E^{(0)}_n + E^{(0)}_m) \lambda + E^{(0)}_n E^{(0)}_m - |V_{nm}|^2 = 0$$

$$\lambda = \frac{E^{(0)}_n + E^{(0)}_m}{2} \pm \frac{1}{2} \sqrt{(E^{(0)}_n - E^{(0)}_m)^2 + 4|V_{nm}|^2}$$

as $E_m \to E_n$, $\lambda \to E_n \pm |V_{nm}| \leq$ perturbed energies separated by $\delta$.
a) \[ \psi_n = \frac{1}{\sqrt{L}} e^{i \frac{2\pi n x}{L}} \]

(b) \[ \pm n \] are even

(c) \[ E^{(i)} = \frac{i}{L} \int e^{i \frac{2\pi n x}{L}} e^{-i \frac{2\pi n x}{L}} \sum_{n=0}^{N-1} V_n \, dx \]

\[ = \sum_{n=0}^{N-1} \int V_n \, dx = 0 \]

\[ V(x) \neq 0 \quad \text{for} \quad 0 < x < \frac{L}{N} \]

\[ = 0 \quad \text{otherwise} \]

\[ V_0(x) = V(x) \]

\[ V_1(x) = V(x - \frac{L}{N}) \neq 0 \]

(d) \[ V_{mn} = \int_0^L e^{-i \frac{2\pi (m-n) x}{L}} \sum_{p=0}^{N-1} V_p(x) \, dx \]

\[ = \sum_{p=0}^{N-1} \int e^{-i \frac{2\pi (m-n) x}{L}} V(x + \frac{L}{N}) \, dx \]

\[ = \sum_{p=0}^{N-1} \left( e^{-i \frac{2\pi (m-n) L}{N}} \int L/N, e^{i \frac{2\pi (n-m) i}{N}} \right) \int_0^{L/N} V(x') \, dx' \]

\[ \text{In} m \]

\[ \frac{2\pi (n-m) i}{L} = 0 \quad \text{unless} \quad n-m = jN \]

\[ \sum_{p=0}^{N-1} e^{i \frac{2\pi j p}{N}} = N \]

in which case the sum is
\[ V_{mn} = \mathcal{N} I_{nm} \quad n - m = jN \quad j = \pm 1, \pm 2, \ldots \]

\( j = 0 \) doesn't work, then \( n = m \) and \( V_{nm} = 0 \)

\[
E^{(2)}_{n} = \sum_{m \neq n} \frac{|V_{mn}|^2}{E_{n}^{(1)} - E_{m}^{(1)}} = \sum_{j} \frac{|V_{n-jN, n}|^2}{E_{n}^{(1)} - E_{n-jN}^{(1)}}
\]

For most values of \( j \) the energy denominator will be large and \( E^{(2)}_{n} \) just makes a small difference in the spectrum, but for \( E_{n}^{(1)} \approx E_{m}^{(1)} \) it can be large. This is the case when

\( n-jN \approx -n \) since \( E_{n} \approx E_{-n} \)

\( 2n \approx jN \), \( n = \frac{jN}{2} \), \( j = 1, 2, \ldots \)

Let \( n = \frac{jN}{2} \neq p \) \( p = \) small integer

\( m = n - jN = \frac{jN}{2} \neq p - jN = -\left(\frac{jN}{2} \pm p\right) \)

\[
E^{(2)}_{\frac{jN}{2} \neq p} = \frac{|V_{-\left(\frac{jN}{2} \pm p\right)}, \frac{jN}{2} = p|^2}{\left| E_{\frac{jN}{2} \neq p} - E_{\left(-\frac{jN}{2} \pm p\right)} \right|^n \left| \frac{m}{n} \right|}
\]
Boltzmann statistics:

\[ P \propto e^{-\frac{\varepsilon_i}{kT}} \quad \text{with} \quad \sum_i e^{-\frac{\varepsilon_i}{kT}} = Z \quad \text{with} \quad \varepsilon_i = -\mu \cdot \vec{B} \]

\[ P(\text{spin}\parallel \vec{H}) = \frac{e^{\frac{\mu B}{kT}}}{e^{-\frac{\mu B}{kT}} + e^{\frac{\mu B}{kT}}} = \frac{1}{1 + e^{-\frac{2\mu B}{kT}}} \]

\[ T = \frac{2\mu B}{k \ln \left( \frac{P}{1-P} \right)} \quad \text{which for} \quad P = 0.75 \]

and \[ B = 3 \times 10^4 \text{ G} \]

yields

\[ T = \frac{2 \times 0.927 \times 10^{-20} \times 3 \times 10^4}{1.38 \times 10^{-16} \ln \left[ \frac{0.75}{0.25} \right]} = 3.7 \text{ K}. \]
Take the origin at \( O \) and the \( z \)-axis in the \( \hat{z} \) direction. The current is steady and confined to the medium; \( \frac{\partial \Phi}{\partial t} = 0 \) so \( \nabla \times \mathbf{J} = 0 \) but \( \mathbf{J} = \sigma \mathbf{E} \) (ohmic medium)

\[
\text{so } \nabla \cdot \mathbf{E} = 0 \quad \text{and } \mathbf{E} = -\nabla \Phi
\]

so \( \nabla^2 \Phi = 0 \)

The potential everywhere in the medium must satisfy Laplace’s Eq. and the boundary conditions.

Consider the potential \( \Phi = -\frac{V}{d} r \cos \theta + \frac{V}{d} \frac{a^2}{r^2} \cos \theta \)

We keep only \( n=1 \) Zonal Harmonics because

a) the first term gives the correct potential at \( z = r \cos \theta = d \), where the second term can be neglected relative to it, since it is \( \frac{a^2}{r^3} \) smaller in magnitude

b) the second term is the only one which can reduce \( \Phi = 0 \) for all \( \theta \)

because the \( P_n(\cos \theta) \) are orthogonal.

The potential shown

1) is a sum of zonal harmonics, so it satisfies \( \nabla^2 \Phi = 0 \)

2) it makes the lump at \( r = a \)

an equipotential \( \Phi(a, \theta) = 0 \)

3) it makes the rest of the left-hand plate, \( |r| > a \), the same equipotential:

\( \Phi(r > a, \pm \frac{\pi}{2}) = 0 \)

4) it gives the correct field (and potential)

\[
\mathbf{E} = -\nabla \Phi = \frac{V \hat{z}}{d} \quad \text{at } z = d
\]

Therefore the uniqueness theorem guarantees it is the solution in the space between the plates.
Therefore, on the bump's surface, \( n = a \),

\[
E_n = -\frac{\partial \phi}{\partial n} \mid_{n=a} = \frac{V}{a} \cos \theta \left(1 + \frac{2a^2}{n^2}\right) \mid_{n=a} = \frac{V}{a} \cos \theta
\]

is the total radial field,

which makes the surface charge density on the bump

\[
\sigma = D_n = \varepsilon E_n = \frac{3 \varepsilon V}{a} \cos \theta
\]

from the dielectric boundary condition \( D_{n1} - D_{n2} = \sigma \).

Now the radial outward force on each surface element \( dS \) is just

\[
dF_n = \frac{1}{2} \sigma E_n \, dS \quad \text{(or \quad } dF_n = \tau \, dS \text{ where } \tau \text{ is the field energy density.)}
\]

The factor of \( \frac{1}{2} \) comes in because we are expressing \( F_n = F_n \, (E_n)_{total} \).

The energy density form is equivalent because

\[
\frac{1}{2} \sigma E_n = \frac{1}{2} \sigma \frac{D_n}{\varepsilon} = \frac{1}{2} \frac{\varepsilon V^2}{\varepsilon} = \frac{D_n^2}{2 \varepsilon} = \tau \quad \text{for a linear dielectric.}
\]

Further, all components of \( dF_n \) will cancel, due to symmetry, except the \( z \)-component.

\[
\int_{0}^{\pi} dF_n \cos \theta \quad \text{the net force on the bump is then}
\]

\[
\frac{F_z}{\tau} = \int_{0}^{\pi} \frac{1}{2} \frac{\varepsilon V^2}{\varepsilon} \cos \theta \, dS \int_{0}^{\pi/2} \frac{9 \pi \varepsilon V^2}{\varepsilon} \, \cos^3 \theta \, d\theta = \frac{9}{2} \pi \varepsilon \left(\frac{V}{a}\right)^2 a^2 \varepsilon \cos \theta \left|_{\theta=0}^{\pi/2}\right. = \frac{9}{4} \pi \varepsilon \left(\frac{V}{a}\right)^2
\]

Note: the bump's projected area is \( \pi a^2 \); \( \frac{1}{2} \varepsilon \left(\frac{V}{a}\right)^2 \) is the energy density with no bump.

So with no bump, that area would experience a force \( F_z = \frac{9}{4} \pi \varepsilon a^2 \varepsilon \left(\frac{V}{a}\right)^2 \)

The force on the bump is \( 4.5 \) times as large!!
a) Between 0 and \( \tau \), the equation which determines the evolution is

\[
i \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + \nabla \psi - \mathbf{F} \cdot \mathbf{r} \psi
\]

where \( \mathbf{F} \cdot \mathbf{r} \) is the work done on the particle by the force.

The first two terms on the right are negligible compared to the large impulse term, so

\[
\frac{1}{\psi} \frac{\partial \psi}{\partial t} = i \frac{\mathbf{F} \cdot \mathbf{r}}{\hbar} \quad (0 < t < \tau)
\]

\[
\psi(\mathbf{r}, \tau) = \psi(\mathbf{r}, 0) e^{i \frac{\mathbf{F} \cdot \mathbf{r}}{\hbar} \tau}
\]

b) Since \( \psi(\mathbf{r}, 0) \) is normalized,

\[
\langle \psi(\mathbf{r}, \tau) \mid \mathbf{p} \mid \psi(\mathbf{r}, \tau) \rangle = \frac{\hbar}{i} \int \psi^*(\mathbf{r}, \tau) \nabla \psi(\mathbf{r}, \tau) \; d\mathbf{r} = \frac{\hbar}{i} \int \psi^*(\mathbf{r}, \tau) \left( \nabla \psi(\mathbf{r}, \tau) + i \frac{\mathbf{A}}{\hbar} \psi(\mathbf{r}, \tau) \right) \; d\mathbf{r}
\]

\[
= \langle \psi(\mathbf{r}, 0) \mid \mathbf{p} \mid \psi(\mathbf{r}, 0) \rangle + \mathbf{A}
\]

\[
\langle \mathbf{F} \rangle = \langle \mathbf{p} \rangle - \langle \mathbf{p} \rangle = \mathbf{A}
\]

This is the same result as one gets in classical mechanics; the change in the particle's (expectation) value of the linear momentum is just equal to the impulse which acts on it.
c) After the impulse, for our one-dimensional oscillator,

\[ \psi(x, \tau) = \psi(x, 0) e^{i \frac{Ax}{\hbar}} \]

where \( \psi(x, 0) = \psi(x) = \left( \frac{m \omega}{\hbar} \right)^{\frac{1}{4}} e^{-\frac{m \omega x^2}{2 \hbar^2}} \) since \( H_0(x) = 1 \)

Now \( \psi(x, \tau) = \sum_n c_n \psi_n(x) \) since the \( \psi_n \) are a complete, orthonormal set.

and \( \omega_n^2 = |c_n|^2 \) where the \( c_n \) are given by

\[
\omega_n = \left( \frac{m \omega}{\hbar} \right)^{\frac{1}{2}} \frac{1}{\sqrt{2^{2n}(n!)}} \int_{-\infty}^{\infty} e^{-\xi^2} H_n(x) e^{i \frac{Ax}{\hbar}} \frac{(\frac{m \omega}{\hbar})^n}{\hbar} \ d\xi
\]

where \( \lambda = \frac{A}{\sqrt{\hbar \omega}} \)

\[
\omega_n = \frac{(i \lambda)^n}{\sqrt{2^{2n}(n!)}} \int_{-\infty}^{\infty} e^{-\xi^2 + i \lambda \xi + \frac{\lambda^2}{2}} d\xi
\]

\[
= \frac{(i \lambda)^n}{\sqrt{2^{2n}(n!)}} \int_{-\infty}^{\infty} e^{-\xi^2 + i \lambda \xi + \frac{\lambda^2}{2}} d\xi
\]

\[
= \frac{(i \lambda)^n}{\sqrt{2^{2n}(n!)}} \left( \frac{\lambda^2}{2} \right)
\]

\[
= \frac{(i \lambda)^n e^{-\frac{\lambda^2}{2}}}{\sqrt{2^{2n}(n!)}}
\]

\[
|c_n|^2 = \frac{\lambda_n^2}{2^{2n}(n!)} = \left( \frac{A^2}{2 \hbar \omega} \right)^n e^{-\frac{A^2}{2 \hbar \omega}} = \frac{\alpha^n e^{-\alpha}}{n!}
\]

Notice that this is just \( \frac{1}{n!} \left( \frac{\hbar \omega}{2mE_0} \right)^n e^{-\frac{\hbar \omega d^2}{2mE_0}} = \frac{1}{n!} \left( \frac{\hbar \omega}{2mE_0} \right)^n e^{-\frac{\hbar \omega d^2}{2mE_0}} \) a Poisson distribution like a modified Boltzmann factor.
\[ \frac{\Delta L}{L} = \frac{A \Delta \phi}{A} = \Delta \phi \]

\[ \Omega_p = \frac{\Delta \phi}{\Delta t} = \frac{\Delta L}{L \Delta t} \]

\[ \tau = \frac{\Delta L}{\Delta t} = L \Omega_p \equiv I \omega \Omega_p. \]

\[ \therefore \Omega_p = \frac{\tau}{I \omega} \]

\[ I = \frac{m R^2}{2} \]

\[ \tau = l mg \cos \theta \]

\[ \Omega_p = \frac{2 l mg \cos \theta}{\mu c R^2 \omega} \]

Maximum value for \( \Omega_p \) will occur at \( \min. \theta \).

\( \min. \theta \) occurs when head of tack is about to touch floor.

\[ \cos(\theta_{min}) = \frac{l}{(l^2 + R^2)^{1/2}} \]

\[ \therefore \Omega_p_{max} = \frac{2 g (l/R)^2}{\omega R \left[1 + (l/R)^2 \right]^{1/2}}. \]
November 12, 1981

Memo to: All Faculty and Messrs. Bell, Gaskill, and Steve Kirby

From: C. W. Drake, Chairman

Subject: Revised M. S. Requirements

The attached suggestions have been made by the Graduate Curriculum Committee. Please consider the ramifications of the proposal and be prepared to discuss it at a forthcoming faculty meeting.

Written comments would be welcomed and would be passed on to the committee. Any changes would not affect presently enrolled graduate students.

CHD:jw
Enclosure
November 10, 1981

TO: C.W. Drake, Chairman

FROM: Graduate Curriculum Committee: K. Krane, V. Madsen, A. Wasserman, F. Bell

SUBJ: M.S. Degree Requirements

In response to the suggestions made at the last faculty meeting, the Graduate Curriculum Committee met on 5 November 1981 to try to remove some ambiguities from the present Comprehensive Examination requirements. These ambiguities were concerned with the program for the M.S. Degree, and in particular involved the requirement that the candidate pass "2 of at most 4 consecutive examinations". Comments were made at the faculty meeting that the 4-exam restriction was ambiguous and arbitrary; the committee concurred in that view. A corollary problem with the present system is that it allows (and even encourages) students who clearly have no hope of passing the exam (at the M.S. level) to continue taking the exam through 3 and even 4 years.

To correct these (and other) deficiencies in the present system, the Committee makes the following proposals for changes in our present system, retaining the schedule, format, and basic requirements of the present system:

(1) All students (M.S. as well as Ph.D.) must pass at the M.S. level at least one exam of their first 3 "official" tries. (A M.S. pass on a "practice" exam will satisfy this requirement.)

(2) All students who meet requirement (1) will be permitted 6 "official" tries (in addition to the 3 "practice" exams normally taken during the first year).

(3) A student will be considered to have satisfied the Comprehensive Exam requirements for the M.S. or Ph.D. by passing 2 exams at the appropriate level. No "carry-over" points will be permitted.

(4) A student who fails to satisfy requirement (1) above has the alternative, as in the present system, to complete the M.S. degree with the thesis option. In order to be permitted to return as a graduate student in the next academic quarter the student must, before the end of the academic quarter in which the last exam was attempted and failed, complete the following steps of the M.S. thesis program: (a) select a thesis project director; (b) select a committee; (c) submit a program to the graduate school; (d) select a thesis project, establish a timetable for its completion, and have both project and timetable approved by the committee. It is strongly recommended that the M.S. requirements be completed within one academic year following the quarter in which the last exam was failed. The student is of course permitted no further attempts at the exam during this period.
(5) To encourage first year students to take advantage of the 3 "practice" attempts, it is proposed to permit first year students to request anonymity. Their identities will not be revealed to the faculty and their scores will not be recorded in their personal files. They will be notified of their scores and of course may count a passing performance toward meeting their degree requirements.