# PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #39

April 4, 1981

## General Instructions

This Comprehensive Examination for Spring 1981 (#39) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

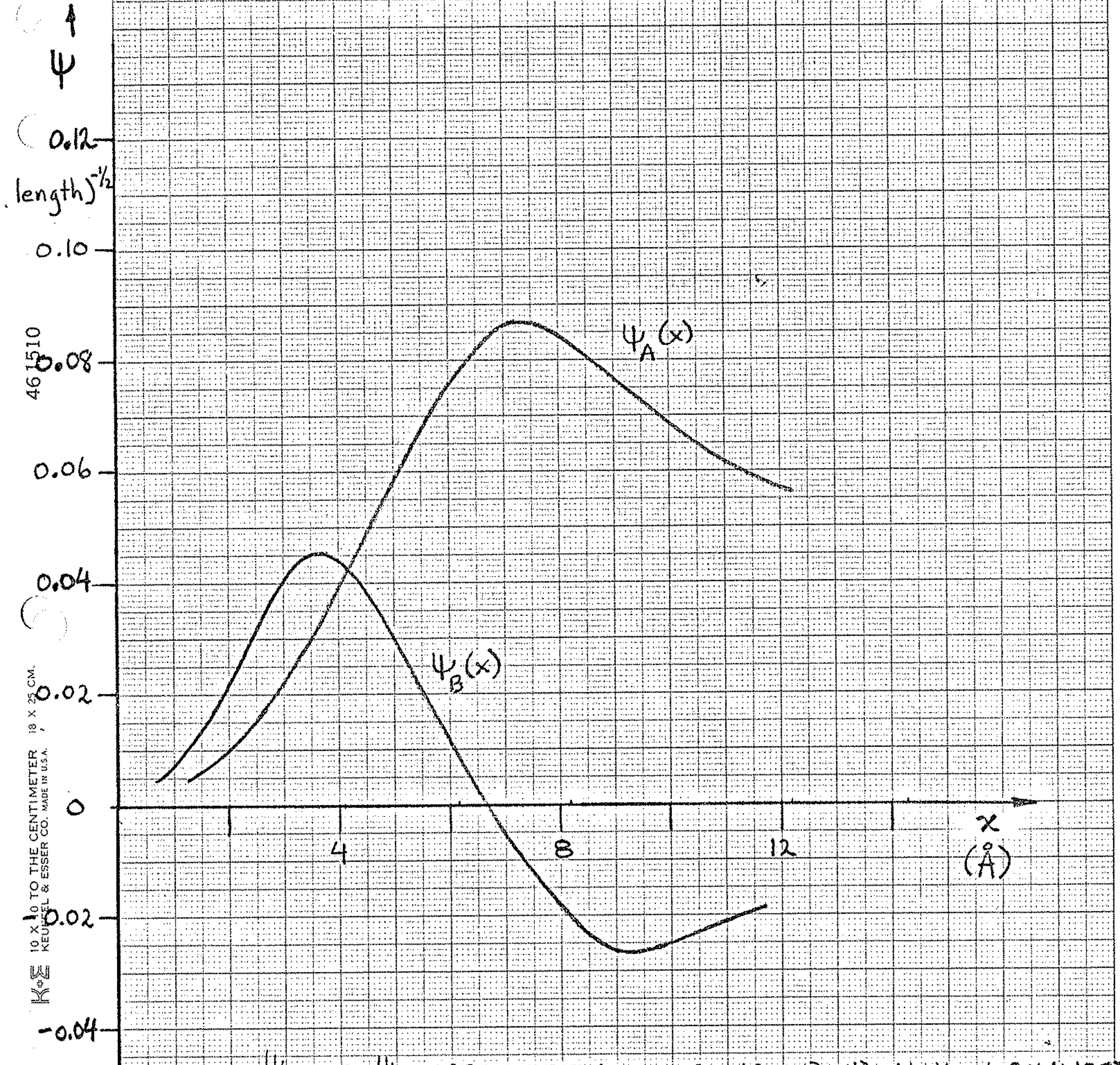
1. A spherically symmetric charge distribution $\rho(r)$ gives rise to the electrical potential

$$\phi(r) = V_0 \frac{e^{-\alpha r}}{\alpha r} \qquad \text{for } a < r < b.$$

(15) (a) From this information, what can you say about $\rho$ for $0 < r < b$?

( 5) (b) What is your answer if $a = 0$, $b = \infty$?

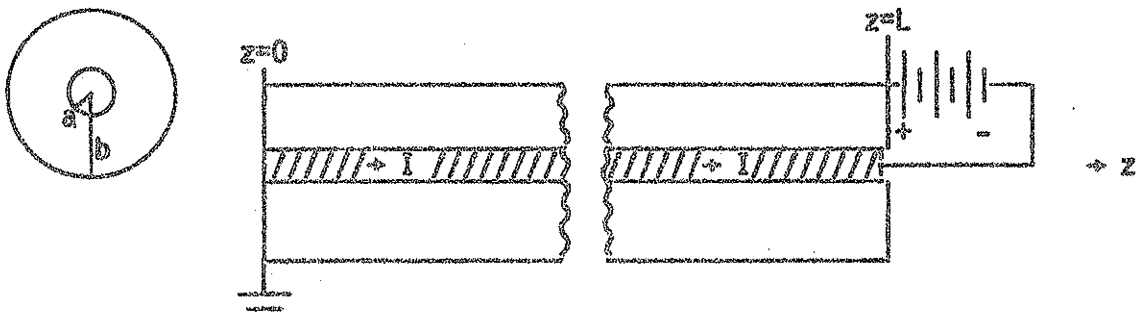
(20) 2. Consider two electrons in a hypothetical <u>one-dimensional</u> atom to be characterized by single particle wavefunctions $\psi_A(x)$ and $\psi_B(x)$ as shown in the accompanying figure. Construct the appropriate two particle wavefunctions for this system and determine the probability densities associated with one electron being located at $x = +3\text{Å}$ simultaneously with the other electron being located at $x = +10\text{Å}$. (The "nucleus" is situated at the origin.)




$\psi_A$ AND $\psi_B$ ARE TO BE ASSUMED AS INDIVIDUALLY NORMALIZED.

`0)     3. Consider a <u>classical</u> system of N non-interacting diatomic molecules enclosed in a container of volume V at temperature T for which the single particle Hamiltonian is $H = \frac{1}{2m}(p_1^{\,2} + p_2^{\,2}) + \frac{1}{2}K\,|\vec{r}_1 - \vec{r}_2|^2$. Determine the mean square molecular diameter $\left\langle |\vec{r}_1 - \vec{r}_2|^2 \right\rangle$ from consideration of the system's partition function. [Hint: Relative and center of mass coordinates would be convenient. Also, $\Gamma(n) = \int_0^\infty u^{n-1}\, e^{-u}\, du = (n-1)\Gamma(n-1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ .]

(20)     4.




A long uniform cylindrical conductor of radius a and length L is completely within a conducting concentric coaxial shield of radius b. The inner conductor is ohmic with resistance per unit length $R_0$ ohms/meter, and carries a <u>constant</u> current I as shown, in the +z direction. Calculate the Poynting Vector due to the static fields in the space between the conductors. Describe the behavior of each of its components as a function of the radial coordinate r, and the longitudinal coordinate z. Hints: The equation to be satisfied is separable, and must not depend on the azimuthal angle θ for reasons of symmetry.

5. The classical description of an assemblage of particles requires them to be distinguishable, and traveling in precisely defined trajectories. That is, there will be no quantum limitations if each particle can be localized by its deBroglie wavelength to within a length that is small compared to the typical separation $s_0$ between the particles, i.e.

$$\frac{\lambda}{s_0} \ll 1 .$$

( 5) (a) Estimate $\lambda$, $s_0$ and $\frac{\lambda}{s_0}$ for a box of He gas at standard temperature and pressure. $m_{He} = 6.6 \times 10^{-27}$ kg.

( 5) (b) Repeat part (a) for a "free electron gas" in copper, assuming one free electron per atom, $A_{Cu} \approx 60$, $\rho_{Cu} \approx 10$ gm/cm$^3$.

( 5) (c) Using the classical equipartition theorem, calculate the molar specific heat of a solid, considering each of the atoms as a harmonic oscillator bound to a fixed position.

( )) (d) The criterion that a classical calculation like (c) be valid is just the requirement that the solid's atoms can be localized, even though vibrating, such that $\Delta p \Delta x \gtrsim \hbar$ is not a significant consideration. That is, we can state the <u>classical behavior validity criterion</u> as $s_0 p_0 \gg \hbar$. Because it is hard (low compressibility), diamond at room temperature has its atoms vibrating with $\omega \approx 10^{14}$ sec$^{-1}$. <u>Using this criterion</u>, decide whether the specific heat you calculated in (c) applies to diamond at room temperature.

6. Under a Lorentz transformation from fixed frame $S$ to moving frame $S'$ the coordinate transforms as

$$x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \quad , \qquad \text{where} \quad x_4 = ict \tag{1}$$

For $S'$ moving along the $+x$ axis

$$\begin{aligned} x' &= \gamma\left(x - \frac{v}{c}\, t\right) \\ t' &= \gamma\left(t - \frac{v}{c^2}\, x\right) \end{aligned} \tag{2}$$

( 2) (a) What does it mean to say that a law of physics is written in covariant form?

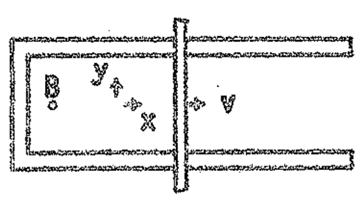
( 2) (b) Why should fundamental physical laws be expressible in covariant form?

( 2) (c) How does the four-vector potential $\vec{A}, i\phi$ transform under the Lorentz transformation?

( 2) (d) Determine the coefficients of transformation $a_{\mu\nu}$ from Eq. (2).

( 4) (e) Interpret the quantities $F_{\mu\nu} = \dfrac{\partial A_\nu}{\partial x_\mu} - \dfrac{\partial A_\mu}{\partial x_\nu}$ .

( 8) (f) Now consider a conducting loop consisting of a sliding wire on a u-shaped wire as shown.




$\vec{B} = B\hat{z}$

The circuit sits in the x-y plane in a magnetic field in the z direction as shown. The sliding wire is moving along the +x axis with velocity v. Determine the electromagnetic field in the reference frame of the moving wire and calculate the force on a charge q, dragged along in the wire, in both the fixed and the moving reference frame in the limit of small v/c. Compare your results.

① In $a<r<b$

$$-4\pi\rho = \nabla^2 \Phi = \frac{V_0}{\alpha}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\left(\frac{e^{-\alpha r}}{r}\right)$$

$$= V_0\,\alpha\,\frac{e^{-\alpha r}}{r}$$

(a) $$\rho = -\frac{V_0\alpha}{4\pi}\,\frac{e^{-\alpha r}}{r} \quad \text{for } a<r<b$$

$$E = -\nabla\Phi = \frac{V_0}{\alpha}\left(\frac{1}{r^2}+\frac{\alpha}{r}\right)e^{-\alpha r}$$

$$\int^r \rho\, d^3r = \oint_{\text{at } r} E\cdot ds = \frac{V_0}{\alpha}\,4\pi(1+\alpha r)\,e^{-\alpha r} \equiv Q(r)$$

at $r=a$ $\quad Q(a) = \frac{4\pi V_0}{\alpha}(1+\alpha a)e^{-\alpha a}$

there is this much positive charge within the sphere at $a$.

[illegible]

(b) If $a \to 0$

$$Q(a) = \frac{4\pi V_0}{\alpha}$$

there is a finite positive charge at $r = 0$

and $\rho = -\frac{V_0 \alpha}{4\pi} \frac{e^{-\alpha r}}{r}$ for $0 < r < \infty$.

[ is $Q(a) < Q(0)$ for $a > 0$

compare $(1+\alpha a) e^{-\alpha a}$ with 1

yes: the exponential falls off faster than $\alpha a$ rises. For small $\alpha a$

$$(1+\alpha a) e^{-\alpha a} \approx (1+\alpha a)\left(1 - \alpha a + \frac{(\alpha a)^2}{2}\right)$$
$$= 1 - \tfrac{1}{2}(\alpha a)^2 < 1$$ ]

② $$\psi_T = \frac{1}{\sqrt{2}}\left[\psi_A(x_1)\,\psi_B(x_2) \pm \psi_A(x_2)\,\psi_B(x_1)\right]\cdot \chi_{\mp}$$

where $\chi_{\mp}$ need not concern us in calculating the spatial probability density; $P(x_1,x_2) = \psi_T^*\psi_T$

$$P(x_1,x_2) = \frac{1}{2}\left|\psi_A(x_1)\psi_B(x_2)\right|^2 + \frac{1}{2}\left|\psi_A(x_2)\psi_B(x_1)\right|^2$$
$$\pm \frac{1}{2}\left\{\psi_A^*(x_1)\psi_B(x_1)\psi_B^*(x_2)\psi_A(x_2) + \psi_B^*(x_1)\psi_A(x_1)\psi_A^*(x_2)\psi_B(x_2)\right\}$$

where we note that expression in $\{\}$ is just 2 times either term in the brackets.

Graph yields $\psi_A(+3) = 0.022$, $\psi_B(+3) = 0.040$
$\psi_A(+10) = 0.068$, $\psi_B(+10) = -0.025$

$$\therefore\ P(+3,+10) = \frac{1}{2}\left|-2.2\times10^{-2}\times 2.5\times10^{-2}\right|^2 + \frac{1}{2}\left|6.8\times10^{-2}\times 4.0\times10^{-2}\right|^2$$
$$\pm \frac{1}{2}\,2\left(2.2\times10^{-2}\times 4.0\times10^{-2}\times(-2.5)\times10^{-2}\times 6.8\times10^{-2}\right)$$
$$= (3.85 \mp 1.50)\times10^{-6}$$

$$P(+3,+10) = 2.35\times10^{-6} \quad \text{(singlet case)}$$
$$= 5.35\times10^{-6} \quad \text{(triplet case)}$$

3 $$Z_N(V,T) = \frac{1}{(2N)!\,h^{6N}}\left[\int e^{-\beta\left\{\frac{p_1^2+p_2^2}{2m} + \frac{K}{2}|\vec{r}_1-\vec{r}_2|^2\right\}} d^3p_1\,d^3p_2\,d^3r_1\,d^3r_2\right]^N$$

which, upon transforming to relative coordinates $\vec{r} = \vec{r}_1 - \vec{r}_2$ and center of mass coordinates $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, becomes

$$Z_N(V,T) = \frac{1}{(2N)!\,h^{6N}}\left[\int e^{-\beta p_1^2/2m} d^3p_1 \int e^{-\beta p_2^2/2m} d^3p_2 \int e^{-\beta \frac{K}{2} r^2} d^3r \int d^3R\right]^N$$

Using definition of the $\Gamma$ function given:

$$Z_N(V,T) = \frac{1}{(2N)!\,h^{6N}}(2\pi m kT)^{3N}\, V^N\left(\frac{2\pi kT}{K}\right)^{3N/2}$$

Now from definition of $Z_N(V,T)$ we see that

$$\left\langle \sum_j |\vec{r}_{1_j} - \cancel{k}\vec{r}_{2_j}|^2\right\rangle = -\frac{2}{\beta}\frac{\partial}{\partial K}\ln Z_N(V,T)$$

but $\left\langle \sum_j |\vec{r}_{1_j} - \vec{r}_{2_j}|^2\right\rangle = N\langle|\vec{r}_1 - \vec{r}_2|^2\rangle$ and so

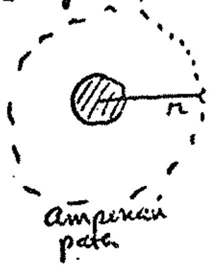
$$N\langle|\vec{r}_1-\vec{r}_2|^2\rangle = -\frac{2}{\beta}\frac{\partial}{\partial K}\ln\left(\frac{2\pi kT}{K}\right)^{\frac{3N}{2}}$$ from our

expression for $Z_N(V,T)$. Thus, finally

$$\langle|\vec{r}_1-\vec{r}_2|^2\rangle = \frac{3kT}{K},$$ a result consistent

with classical equipartition theorem.

(4) We need to determine E and H to calculate
In this region, Applying Ampere's Circuital Law gives us H immediately




$$\oint \vec{H}\cdot d\vec{l} = I$$

$$H_\theta \cdot 2\pi r = I$$

$$\underline{\underline{H_\theta = \frac{I}{2\pi r}}}.$$

However, the electric field has to be determined by solving Laplace's Eq. for the potential, in cylindrical coordinates, $\varphi(r,z)$.

We have the following boundary conditions:

$\underline{\underline{r=a}}$: $\varphi(a) = -E_{in}\, z$ $\left\{\begin{array}{l}\text{It does not surprise us that the potential on the wire is linear in } z. \\ \text{This insures the continuity of the tangential component} \\ \text{of } E \text{ at the boundary; } E_z = -\frac{\partial\varphi}{\partial z} = E_{in} \text{ for the ohmic conductor}\end{array}\right.$

$\underline{\underline{r=b}}$: $E_z(b) = -\frac{\partial\varphi}{\partial z}\Big|_b = 0$ $\left\{\begin{array}{l}\text{this is because there can be no tangential field on the} \\ \text{outer conductor; it is really a Faraday shield, and any} \\ \text{charges flowing in it cannot produce any effect \underline{within}.}\end{array}\right.$

Since the solution cannot depend on $\vartheta$, due to symmetry, Laplace's Eq is.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0$$

we try $\varphi(r,z) = R(r)\, Z(z)$ and then divide by $\varphi$

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

Since each side is a function of only one variable, a solution valid for all $r, z$ requires that each be equal to the same constant, say $m$.

$\underline{\text{If } m>0}$ $\quad \frac{d^2Z}{dz^2} + mZ = 0$ has soln $Z = Ae^{i\sqrt{m}\,z}$. This oscillatory soln violates the boundary condition at $r=a$ ($\varphi \propto z$)

$\underline{\text{If } m<0}$ $\quad \frac{d^2Z}{dz^2} + mZ = 0$ has sol'n $Z = Ae^{-\sqrt{m}\,z}$ which is also unacceptable.

So the only possibility is $\underline{m=0}$ $\quad \frac{d^2Z}{dz^2} = 0$ has solution $Z = A_1 z + B_1$

and $\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$ has sol'n $\quad R = \alpha \ln r + \beta$

so $\quad \varphi = (A_1 z + B_1)(\alpha \ln r + \beta) = A z(\ln r + B)$

We now apply the boundary conditions: Since the current is constant, $\frac{\partial \vec{A}}{\partial t} = 0$ and so

$\underline{r = a}$ $\quad E_z = -\frac{\partial \varphi}{\partial z}\Big|_{r=a} = -A(\ln a + B) = E_{in}$ $\quad$ since $E_{1t} = E_{2t}$ at the boundary

$\underline{r = b}$, $\quad E_z = -\frac{\partial \varphi}{\partial z}\Big|_{r=b} = -A(\ln b + B) = 0$

$$\text{so} \quad \underline{B = -\ln b} \quad , \quad A = -\frac{E_{in}}{\ln\left(\frac{a}{b}\right)}$$

and $\underline{\underline{\varphi = -\frac{E_{in}\, z}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{r}{b}\right)}}$ $\quad$ giving $\begin{cases} E_z = -\frac{\partial \varphi}{\partial z} = +\frac{E_{in}}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{r}{b}\right) \\ E_r = -\frac{\partial \varphi}{\partial r} = +\frac{E_{in}\, z}{\ln\left(\frac{a}{b}\right)} \frac{1}{r} \end{cases}$

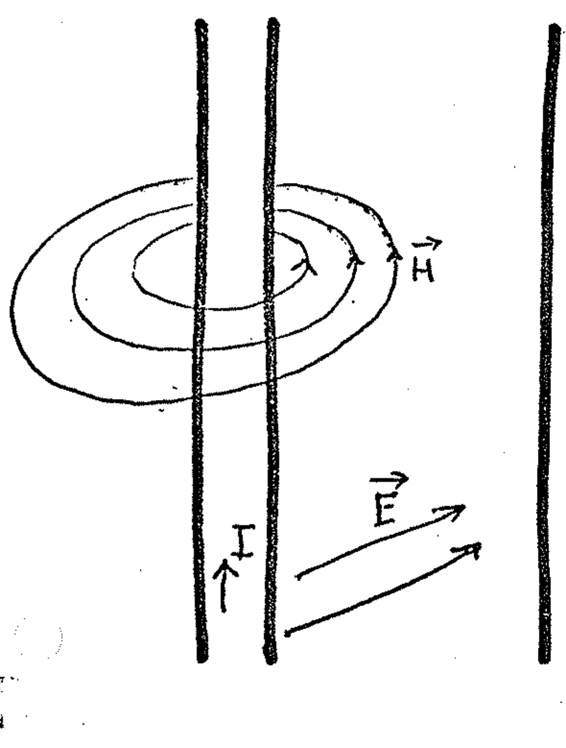
then since $E_{in} = IR_0$

$$S_r = -E_z H_\theta = \frac{I^2 R_0}{2\pi \ln\left(\frac{a}{b}\right)} \left[\frac{\ln\left(\frac{r}{b}\right)}{r}\right](-\hat{r})$$

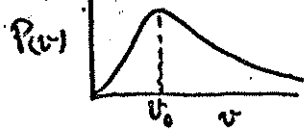
directed inward, to supply $I^2R$ losses
same at all $z$, decreases with $r$
like $\frac{1}{r}\ln\left(\frac{r}{b}\right)$

$$\underline{\underline{S_z = +E_r H_\theta = \frac{I^2 R_0\, z}{2\pi \ln\left(\frac{a}{b}\right)} \frac{1}{r^2} \; (+\hat{k})}}$$

decreases sharply away from the central wire, as $\frac{1}{r^2}$. Carries energy $\underline{\text{down}}$ the wire, which acts as a guide. Increases linearly with $z$.



a) For a classical (Maxwellian) distribution of particles in a box at temperature $T$,




$v_0 = \sqrt{\frac{2kT}{m}}$ so $p = mv = \sqrt{2mkT}$ and $\lambda = \frac{h}{p} \approx \frac{h}{\sqrt{2mkT}}$

$$\lambda \approx \frac{6.6\times10^{-34}\ \text{joule}\cdot\text{sec}}{\sqrt{2\times6.6\times10^{-27}\,\text{kg}\times1.38\times10^{-23}\ \frac{\text{joule}}{{}^\circ\text{K}}\times300^\circ\text{K}}} = \underline{\underline{9.1\times10^{-11}\,\text{m} = 0.91\ \text{A}^\circ}}$$

$S_0^3 = \frac{1}{n}$ so $S_0 = \left(\frac{1}{n}\right)^{1/3}$ but $p = nkT$ so $\frac{1}{n} = \frac{kT}{p}$ and $S_0 = \left(\frac{kT}{p}\right)^{1/3} = \left(\frac{1.38\times10^{-23}\times300^\circ\text{K}}{1\times10^5\ \frac{\text{N}}{\text{m}^2}}\right)^{1/3}$

$$\underline{\underline{S_0 = 3.4\times10^{-9}\,\text{m} = 34\ \text{A}^\circ}}$$

$$\underline{\underline{\left(\frac{\lambda}{S_0}\right)_{He} = \frac{0.91}{34} = 0.027 \ll 1}}$$

So the classical treatment of the gas at STP is justified

---

b) $\frac{m_{electron}}{m_{He}} \approx \frac{1}{4}\times\frac{1}{1800} = \frac{1}{7200}$ So at room temperature, $\lambda_{electron} = \sqrt{7200}\,\lambda_{He} = 85\,\lambda_{He} = \underline{\underline{76\ \text{A}^\circ}}$

$$S_0 = \left(\frac{A}{\rho N_0}\right)^{1/3} \approx \left(\frac{60}{10\times6\times10^{23}}\right)^{1/3} = \left(10\times10^{-24}\right)^{1/3} = \underline{\underline{2\ \text{A}^\circ}}$$

$$\underline{\underline{\left(\frac{\lambda}{S_0}\right)_{electron} = \frac{76}{2} = 38 > 1}}$$

So the electrons form a dense gas in Cu; they cannot be treated classically and the Pauli Exclusion principle will determine most properties.

---

c) The classical equipartition theorem says that <u>each quadratic term</u> in the energy of a system of particles will have an average value of $\frac{1}{2}kT$.

Each atom has energy $\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$ where $\varepsilon_x = \frac{p_x^2}{2m} + \frac{1}{2}ax^2$ where the vibration frequency is $\omega = \sqrt{\frac{a}{m}}$

so $\bar{\varepsilon} = \bar{\varepsilon}_x + \bar{\varepsilon}_y + \bar{\varepsilon}_z = 3\left(\frac{1}{2}kT + \frac{1}{2}kT\right) = 3kT$

For one mole with $N_0$ atoms, $\bar{E} = 3N_0kT = 3RT$

so $C_v \equiv \left(\frac{\partial\bar{E}}{\partial T}\right)_v = \underline{\underline{3R}} = 3\left(8.3\ \frac{\text{joule}}{\text{mole}\ {}^\circ\text{K}}\right) = \underline{\underline{25\ \frac{\text{joules}}{\text{mole}\ {}^\circ\text{K}}}}$ for all classical solids

---

d) $\frac{\overline{p_x^2}}{2m} = \frac{1}{2}kT$ so $p_0 \approx \sqrt{\overline{p_x^2}} = \sqrt{mkT}$

$\frac{1}{2}a\overline{x^2} = \frac{1}{2}kT$, $\quad s_0 \approx \sqrt{\overline{x^2}} = \sqrt{\frac{kT}{a}}$

$$\left.\begin{array}{l} s_0\cdot p_0 \approx kT\sqrt{\frac{m}{a}} = \frac{kT}{\omega} \gg \hbar \\ \text{or } \underline{\frac{kT}{\hbar\omega} \gg 1} \end{array}\right\} \text{for classical validity}$$

for diamond atoms $\underline{\underline{\frac{kT}{\hbar\omega}}} \approx \frac{1.38\times10^{-23}\times300}{10^{-34}\times10^{14}} = \underline{\underline{0.4 < 1}}$

So we do not expect $C_v = 3R$.

In fact $(C_v)_{diamond} \approx 6\ \frac{\text{joule}}{\text{mole}\ {}^\circ\text{K}}$

(6)

(a) It transforms as a Lorentz tensor.

(b) The principle of relativity states that physical laws are ~~invariant~~ the same in all reference frames. This is assured only if the laws are or can be written in covariant form.

(c) $A'_\mu = \sum_\nu a_{\mu\nu} A_\nu$

(d) $a_{11} = \gamma \quad a_{14} = +i\gamma \frac{v}{c^2} t \qquad a_{22} = 1 = a_{33}$

$a_{44} = \gamma \quad a_{41} = -\frac{v\gamma i}{c}$

$x' = \gamma(x + \frac{v}{c^2} i \, \underline{ict})$

$ict' = \gamma(ict - \frac{iv}{c} x)$

(e) $F_{12} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = (\nabla \times A)_3 = B_z$

$F_{41} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} = \frac{\partial A_1}{\partial ict} - \frac{\partial i\varphi}{\partial x_1} = -i\left(\frac{1}{c}\frac{\partial A_1}{\partial t} + \frac{\partial \varphi}{\partial x_1}\right) = +iE_1$

$F_{42} = +iE_2$

$F_{43} = +iE_3$

$F_{13} = \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} = (\nabla \times A)_{13} = -B_2$

$F_{23} = +B_1, \quad F_{12} = B_3$

(f) Only $B_z$ has a value so $F_{12} = -F_{21} = B_z \neq 0$

$F'_{\mu\nu} = \sum a_{\mu\mu'} a_{\nu\nu'} F_{\mu'\nu'} = a_{\mu 2} a_{\nu 1} F_{21} + a_{\mu 1} a_{\nu 2} F_{12}$
$= -B_z (a_{\mu 2} a_{\nu 1} - a_{\mu 1} a_{\nu 2})$

As a componen

Since $\mu = 2$ or $\nu = 2$ appears, either $\mu$ or $\nu$ must be 2

$F'_{12} = B'_z = \cancel{a_{11}, = \gamma B_z} \; -B_z(-a_{11} a_{22}) = \gamma B_z$

$F_{23} = -B_z(a_{22} a_{31} - a_{21} a_{32}) = 0$

$F_{14} = iE_3 = 0, \quad F_{24} = -iE_2 = -B_z(a_{22} a_{41} - a_{21} a_{42}) = -B_z\left(-\frac{v\gamma i}{c}\right)$

$E_2 = -B_z \frac{v}{c} \gamma \quad \approx -B_z \frac{v}{c}$

Original force was $q \cdot \frac{\vec{v} \times \vec{B}}{c} = \frac{(v\hat{x}) \times (B\hat{z})}{c} = -\frac{vB}{c}\hat{y}$

In trans frame $E_2 = -B \cdot \frac{v}{c}$