PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #38

January 10, 1981

General Instructions

This comprehensive Examination for ~~Fall~~ Winter 1981 (#38) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

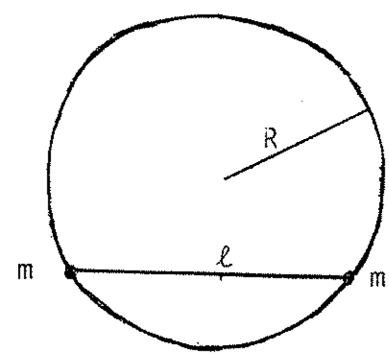
If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

1. Two particles of equal mass m are connected by a massless, rigid rod of length ℓ. The particles are constrained to move in a smooth (frictionless) vertical circle of radius R near the earth's surface.

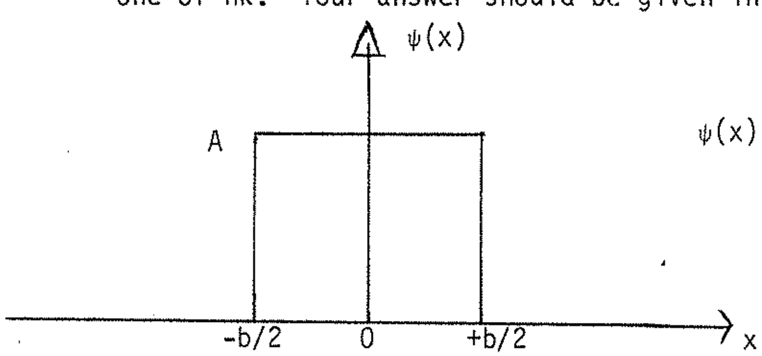
(10) (a) Write the Lagrangian equations of motion describing this specific problem.

(5) (b) Determine an expression for the period of small oscillations from the equilibrium position.

(5) (c) Show that your result converges to that of the simple pendulum for $\ell \to 0$.



(20) 2. Given that a free particle is described by <u>the wave function $\psi(x)$ shown below</u>, determine the ratio of the probability of finding the particle in a momentum state of $2\hbar k$ to that of finding it in one of $\hbar k$. Your answer should be given in terms of k and b alone.



$$\psi(x) = \begin{cases} A \ ; & -b/2 \leq x \leq +b/2 \\ 0 \ ; & x < -b/2,\ x > +b/2 \end{cases}$$

3. In a rather crude approximation, scattering of a neutron from a nucleus can be described in terms of potential scattering from a complex square-well potential

$$V = \begin{cases} -V_o - iW_o & r < R \\ 0 & r > R \end{cases}$$

Recall that the scattering amplitude is

$$f = k^{-1} \sum_{\ell} e^{i\delta_\ell} \sin \delta_\ell \sqrt{4\pi(2\ell + 1)}\, Y_\ell^{o} (\theta , \phi)$$

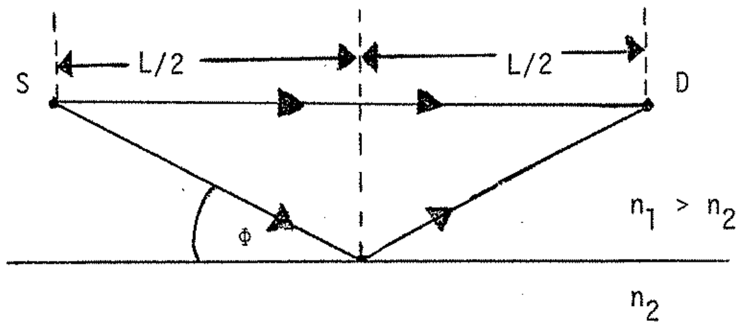
(3) (a) What is the differential scattering cross section in terms of the phase shifts δ_ℓ ?

(4) (b) Integrate the result of part (a) to obtain the integrated cross section in terms of δ_ℓ.

(7) (c) Set up equations from which the phase shifts δ_ℓ for parts (a) and (b) can be determined for the square well. Identify parameters you have used. For this purpose use "ρ-multiplied" spherical Bessel functions $J_\ell(\rho) = \rho j_\ell(\rho)$, $N_\ell(\rho) = \rho n_\ell(\rho)$, $H_\ell(\rho) = \rho h_\ell(\rho)$.

(6) (d) Now consider a particle of low energy so $kR \ll 1$ in a very deep potential well. Show that at most energies the scattering is the same as that from a hard sphere of radius R ($V = + \infty$ for $r < R$).

(20) 4.



Light from a coherent monochromatic (wavelength λ) source S traverses the two paths shown in the figure, arriving at the detector D. The media have refractive indices as indicated. Show that the smallest angle Φ at which a maximum in the intensity at D occurs is given by $\sqrt{\lambda/L}$, where $\lambda << L$.

(20) 5. The gyromagnetic ratio of a current distribution is defined as the magnitude of the ratio of its magnetic dipole moment to its angular momentum $G \equiv | \vec{m}/\vec{\ell} |$. Calculate G for a sphere of mass M, radius R and charge Q which is rotating at constant angular velocity $\vec{\omega} = \omega\hat{k}$ about a fixed diameter. Assume that, while the mass of the sphere is uniform throughout its volume, the charge is distributed uniformly on its surface. (Recall that the moment of inertia for a sphere of radius R and mass M about its diameter is given by $I = 2/5\ MR^2$).

6. A particle is bound by a 3-dimensional harmonic oscillator potential

$$V_0 = \tfrac{1}{2}\mu\omega^2 r^2 = \tfrac{1}{2}\mu\omega^2(x^2 + y^2 + z^2)$$

(5) (a) Find the degeneracies, parities, and energies (in units of $\hbar\omega$) of the ground and first two excited states. How many quantum states are there within these energies?

(2) (b) Determine the first-order energy shift in the first excited energy state from part (a) due to an electric-field perturbation $\Delta V = -zeE_0$

(6) (c) Now consider a perturbation of the system of part (a) of the quadrupole form

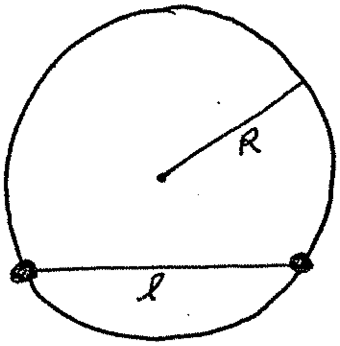
$$\Delta V = \mu\omega^2\eta^2 P_2(\cos\theta) = \tfrac{1}{2}\mu\omega^2\eta^2(2z^2 - x^2 - y^2)$$

Determine the degeneracies and energies in units of $\hbar\omega$ of all the quantum states of part (a).

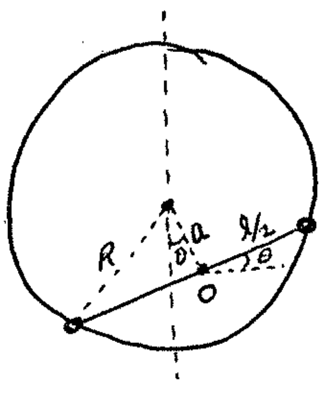
(2) (d) Are angular momentum and parity conserved by the perturbation of part (c) ?

(5) (e) Sketch side by side the spectra of parts (a) and (c) to leading order in η^2, labeling each state with its degeneracy.

Graduate - Dynamics



Two particles of equal mass are connected by a rigid rod of length l. The particles are constrained to move in a smooth (frictionless) vertical circle of radius R near the earth's surface. Write the Lagrangian equations of motion, and determine the period of small oscillations from the equilibrium position. Neglect the rod's mass.



We must consider the motion of the CM, at O, and the system's rotation about O.

$$T = T_{CM} + T_{\text{about } O}$$

$$= \frac{1}{2}(2ma^2)\dot{\theta}^2 + \frac{1}{2} I_o \dot{\theta}^2$$

but $I_o = 2m\left(\frac{\ell}{2}\right)^2$

$$= \frac{1}{2}(2ma^2)\dot{\vartheta}^2 + \frac{m\ell^2}{4}\dot{\theta}^2$$

so we have only one degree of freedom.

$$= mR^2\dot{\vartheta}^2 - \cancel{\frac{m\ell^2}{4}\dot{\theta}^2} + \cancel{\frac{m\ell^2}{4}\dot{\theta}^2}$$

since $a^2 = R^2 - \frac{\ell^2}{4}$

so $T = mR^2\dot{\vartheta}^2$

and $V = (2m)\,g\,a(1-\cos\vartheta)$

so $L = T - V = mR^2\dot{\vartheta}^2 - 2mga(1-\cos\theta)$

$$\begin{cases} \dfrac{\partial L}{\partial \dot{\theta}} = 2mR^2\dot{\theta} \\ \dfrac{\partial L}{\partial \theta} = -2mga\sin\theta \end{cases}$$

then $\boxed{\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{\partial L}{\partial q} = 0}$ becomes

$$\cancel{2m}R^2\ddot{\theta} + \cancel{2m}ga\sin\theta = 0$$

$$\ddot{\vartheta} + \frac{ga}{R^2}\sin\theta = 0$$

$a = \sqrt{R^2 - \frac{\ell^2}{4}}$

$\frac{a}{R^2} = \frac{\sqrt{1 - \frac{\ell^2}{4R^2}}}{R}$

$$\ddot{\theta} + \frac{g}{R}\sqrt{1 - \frac{\ell^2}{4R^2}}\,\sin\theta = 0$$

but for small oscillations, $\sin\theta \approx \theta$ $\quad \ddot{\theta} + \omega^2\theta = 0$

gives

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g\sqrt{1-\frac{\ell^2}{4R^2}}}{R}}$$

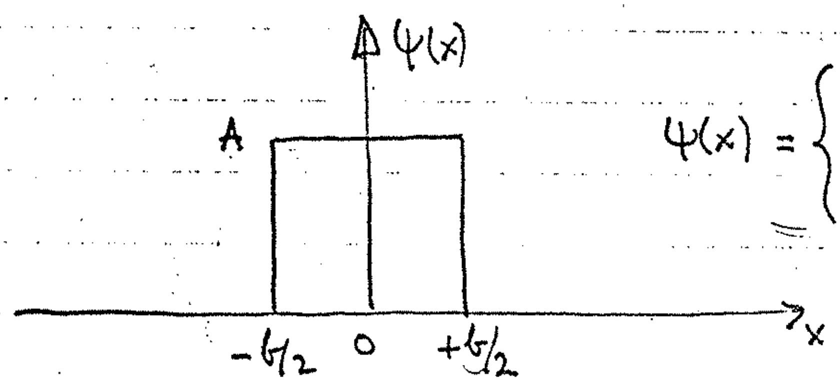
or

$$\tau = 2\pi\sqrt{\frac{R}{g\sqrt{1-\frac{\ell^2}{4R^2}}}}$$

note: 1) τ is independent of m

2) As $\ell \to 0$, we get the correct result for the simple pendulum, as we should.

Given that a free particle is described by the wave function $\psi(x)$ shown below, determine the ratio of the probabilities of finding the particle in a momentum state of $2\hbar k$ to that of finding it in one of $\hbar k$. Your answer should be given in terms of k and b alone.



$$\psi(x) = \begin{cases} A \ ; & -b/2 \le x \le +b/2 \\ 0 \ ; & x < -\frac{b}{2},\ x > +\frac{b}{2} \end{cases}$$

$$B_k = \langle k|\psi\rangle = \frac{1}{\sqrt{2\pi}} \int \psi(x)\, e^{-ikx}\, dx$$

$$= \frac{A}{\sqrt{2\pi}} \int_{-\frac{b}{2}}^{+b/2} e^{-ikx}\, dx = \frac{A}{\sqrt{2\pi}} \frac{\sin(kb/2)}{k/2}$$

$$|B_k|^2 = \frac{A^2 b^2}{2\pi} \left[\frac{\sin(kb/2)}{kb/2} \right]^2$$

and $$|B_{2k}|^2 = \frac{A^2 b^2}{2\pi} \left[\frac{\sin(kb)}{kb} \right]^2$$

$$\therefore \quad \frac{|B_{2k}|^2}{|B_k|^2} = \frac{\sin^2(kb)}{4 \sin^2(kb/2)} = \cos^2\left(\frac{kb}{2}\right).$$

A. (a) $\dfrac{d\sigma}{d\Omega} = |f|^2 = \dfrac{1}{k^2}\Big|\sum_l e^{i\delta_l}\sin\delta_l\sqrt{4\pi(2l+1)}\;Y_l^0(\theta,\phi)\Big|$

(b) $\sigma = \int d\Omega\,\dfrac{d\sigma}{d\Omega} = \dfrac{4\pi}{k^2}\sum_l \sin^2\delta_l\,(2l+1)$

(c) 

Inside $\qquad$ outside

$C\,j_l(Kr) \qquad B\,h_l^{(1)}(kr) + A\,h_l^{(2)}(kr)$

$$K\frac{j_l'(Ka)}{j_l(Ka)} = k\,\frac{e^{2i\delta_l}h_l'^{(1)}(ka) + h_l'^{(2)}(ka)}{e^{2i\delta_l}h_l^{(1)}(ka) + h_l^{(2)}(ka)}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad K = \sqrt{\frac{2m(E+V_0+iW_0)}{\hbar^2}}$$

(d) $$Ka\cot Ka = ika\,\frac{e^{2i\delta_0}e^{ika} + e^{-ika}}{e^{2i\delta_0}e^{ika} - e^{-ika}}$$

Let $$\frac{\gamma}{i} = \frac{K\cot Ka}{ik} = \frac{e^{2i\delta_0}e^{ika} + e^{-ika}}{e^{2i\delta_0}e^{ika} - e^{-ika}}$$

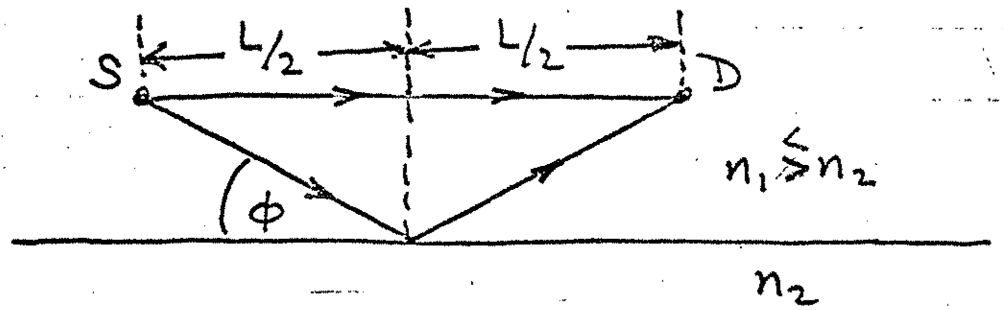
$$e^{2i\delta_0} = -e^{-2ika}\,\frac{1-\gamma/i}{1-\gamma/i} = e^{-2ika}\,\frac{\gamma-i}{\gamma+i}$$

when $K\cot Ka$ is large $e^{2i\delta_0} \approx e^{-2ika}$

whether V_0 is $\pm$; for $V_0 \to +\infty$ we have hard-sphere scatt

or $$\gamma = \frac{Ka\cot Ka}{ka} = \frac{ka}{ka}\,\frac{\cos ka + \tan\delta_l \sin ka}{\sin ka - \tan\delta_l\cos ka}$$

$$\tan\delta_l = \frac{\gamma\sin ka - \cos ka}{\gamma\cos ka + \sin ka}$$



Light from a coherent monochromatic (wavelength λ) ~~light~~ source S traverses the two paths shown in the figure, arriving at the detector D. The media have refractive indices as indicated. Show that the smallest angle ϕ at which a maximum in the intensity at D occurs is given by $\sqrt{\frac{\lambda}{L}}$, where $\lambda \ll L$.

$$\Delta\phi_{TOTAL} = \Delta\phi_{\text{path diff.}} + \pi_{\text{at reflection}}$$

$$= \frac{2\pi}{\lambda}(\Delta \ell) + \pi$$

$$\therefore\ \Delta\phi_T = \frac{2\pi L}{\lambda}\left(\frac{1}{\cos\phi} - 1\right) + \pi = 2\pi \quad \text{for } 1^{st} \text{ max.}$$

$$\text{or } \Delta\phi_T = \frac{2\pi L}{\lambda}\left(\frac{1}{\cos\phi} - 1\right) - \pi = 0 \quad \text{" " "}$$

either way $\dfrac{1}{\cos\phi} = 1 + \dfrac{\lambda}{2L}$

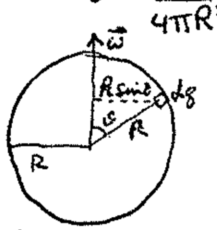
$$\cos\phi \simeq 1 - \frac{\lambda}{2L} \qquad \lambda \ll L$$

But $\cos\phi \simeq 1 - \frac{\phi^2}{2}$, and so $\phi \simeq \sqrt{\frac{\lambda}{L}}$.

Undergrad E+M

The gyromagnetic ratio (dipole) of a current distribution is defined as the magnitude of the ratio of its magnetic moment to its angular momentum: $G \equiv \left|\frac{\vec{m}}{\vec{\ell}}\right|$. Calculate G for a sphere of mass M, radius R and charge Q which is rotating at constant angular velocity $\vec{\omega} = \omega \hat{k}$ about a fixed diameter. Assume that while the mass of the sphere is uniform throughout its volume, the charge is distrubuted unformly on its surface.

$\sigma = \frac{Q}{4\pi R^2}$



$da = R^2 \sin\theta \, d\theta \, d\varphi$

The contribution of the surface element da is

$$d\vec{m} = A \, dI \, \hat{k} = \pi (R\sin\theta)^2 \left(\frac{dq}{T}\right) \hat{k} = \pi (R\sin\theta)^2 \left(\frac{\omega}{2\pi}\right) \sigma \, da \, \hat{k}$$

$$\text{so } \vec{m} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \; \frac{\pi R^2 \omega \sigma}{2\pi} R^2 \sin^2\theta (\sin\theta) \; \hat{k}$$

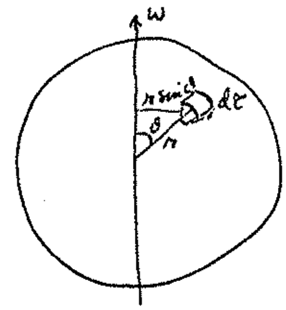
$$= 2\pi \cdot \frac{\pi R^2 \omega \sigma}{2\pi} \int_0^{\pi} \sin^3\theta \, d\theta \; \hat{k}$$

$$\left\{ \begin{aligned} \int_0^{\pi} \sin^3\theta \, d\theta &= \int_0^{\pi} (1-\cos^2\theta) \sin\theta \, d\theta \\ &= \int_{-1}^{1} (1-x^2) dx = 2 - 2/3 = \underline{\underline{4/3}} \end{aligned} \right.$$

$$= \pi R^4 \omega \sigma \; \frac{4}{3} = \frac{4\pi R^4}{3} \omega \frac{Q}{4\pi R^2} \hat{k}$$

$$\underline{\underline{\vec{m} = \frac{1}{3} Q \omega R^2 \hat{k}}}$$

$\rho = \frac{M}{\frac{4}{3}\pi R^3}$; the contribution of the ~~volume~~ element $d\tau$ is



$$d\vec{\ell} = \vec{r} \times \vec{v} \, dm \qquad \left\{ \begin{aligned} \vec{r} \times \vec{v} &= (r\sin\theta) v \; \hat{k} \\ &= (r\sin\theta)(r\sin\theta)\omega \; \hat{k} \\ dm &= \rho \, d\tau = \rho \, r^2 \sin\theta \, dr \, d\theta \, d\varphi \end{aligned} \right.$$

we could give them

$I_0 = 2/5 \, MR^2$

$$= r^2 \sin^2\theta \, \omega \rho \, r^2 \sin\theta \, dr \, d\theta \, d\varphi \; \hat{k}$$

$$\text{So } \vec{\ell} = \omega\rho \int_0^{2\pi} d\varphi \int_0^{R} r^4 dr \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \omega\rho \, (2\pi) \left(\frac{R^5}{5}\right) \left(4/3\right)$$

$$= \omega \frac{M}{\frac{4}{3}\pi R^3} \cdot 2\pi \frac{R^5}{5} \frac{4}{3} = \underline{\underline{\frac{2}{5} M \omega R^2 \; \hat{k}}}$$

and $$G = \frac{Q \omega R^2}{3} \cdot \frac{5}{2 M \omega R^2} = \underline{\underline{\frac{5}{6} \frac{Q}{M}}}$$

B soln Rollin

754-4440

(a) $V = \frac{1}{2}\mu\omega^2 (x^2+y^2+z^2)$

Separation of variables gives 3 linear oscillators $\psi = \alpha_{n_x}(x)\,\beta_{n_y}(y)\,\gamma_{n_z}(z)$

$$E = \hbar\omega\left(n_x+\tfrac{1}{2}+n_y+\tfrac{1}{2}+n_z+\tfrac{1}{2}\right)$$

$$= \hbar\omega(n+3/2), \quad n = n_x+n_y+n_z$$

$$\text{parity} = P = P_x P_y P_z \qquad P_x = (-1)^{n_x}$$

n_x	n_y	n_z	n	$E/\hbar\omega$	degen	parity
0	0	0	0	3/2	1	1
1	0	0	1	5/2	3	−1
0	1	0	1	5/2		−1
0	0	1	1	5/2		−1
2	0	0	2	7/2	6	1
0	2	0	2	7/2		1
0	0	2	2	7/2		1
1	1	0	2	7/2		1
1	0	1	2	7/2		1
0	1	1	2	7/2		1

(b) $\Delta E = \int \psi^* \Delta V \psi \, d\tau = 0$ since ΔV is odd and ψ is even or odd for each energy state

(c) $$V = \frac{1}{2}\mu\omega^2\left[r^2 + \eta^2\underbrace{(3z^2-r^2)}_{2z^2-x^2-y^2}\right] = \frac{1}{2}\mu\omega^2\left[(1-\eta^2)x^2+(1-\eta^2)y^2 + (1+2\eta^2)z^2\right]$$

again one may separate variables as in (a) but now $\frac{1}{2}\mu\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$, where

B soln 2

$$\omega_x = \omega_y = \omega\sqrt{1-\eta^2}$$

$$\omega_z = \omega\sqrt{1+2\eta^2}$$

and $$E = \hbar\left[\omega_x\left(n_x+\tfrac{1}{2}\right) + \omega_y\left(n_y+\tfrac{1}{2}\right) + \omega_z\left(n_z+\tfrac{1}{2}\right)\right]$$

$$= \hbar\omega\left[\sqrt{1-\eta^2}\,(n_x+n_y+1) + \sqrt{1+2\eta^2}\left(n_z+\tfrac{1}{2}\right)\right.$$

the

n_x	n_y	n_z	$E/\hbar\omega$		degen.
0	0	0	$\sqrt{1-\eta^2}+\frac{1}{2}\sqrt{1+2\eta^2}$ $1-\frac{1}{2}\eta^2$ $\frac{1}{2}(1+\eta^2)$	$\approx 3/2$	1
1	0	0	$2\sqrt{1-\eta^2}+\frac{1}{2}\sqrt{1+2\eta^2}$ $1-\frac{1}{2}\eta^2$ $(1+\eta^2)$	$\approx \frac{5}{2}-\frac{1}{2}\eta^2$	2
0	1	0	" " "		
0	0	1	$\sqrt{1-\eta^2}+\frac{3}{2}\sqrt{1+2\eta^2}$ $1-\frac{1}{2}\eta^2+\frac{3}{2}(1+\eta^2)$	$\frac{5}{2}+\eta^2$	1
2	0	0	$3\sqrt{1-\eta^2}+\frac{1}{2}\sqrt{1+2\eta^2}$ $1-\frac{1}{2}\eta^2$ $\frac{1}{2}(1+\eta^2)$	$\frac{7}{2}-\eta^2$	3
0	2	0	" "		
0	0	2	$\sqrt{1-\eta^2}+5/2\sqrt{1+2\eta^2}$ $1-\frac{1}{2}\eta^2+5/2(1+\eta^2)$	$\frac{7}{2}+2\eta^2$	1
1	1	0	$3\sqrt{1-\eta^2}+1/2\sqrt{1+2\eta^2}$ $3(1-\frac{1}{2}\eta^2)+\frac{1}{2}(1+\eta^2)$	$\frac{7}{2}-\eta^2$	
1	0	1	$2\sqrt{1-\eta^2}+\frac{3}{2}\sqrt{1+2\eta^2}$ $2(1-\frac{1}{2}\eta^2)+\frac{3}{2}(1+\eta^2)$	$\frac{7}{2}+\frac{1}{2}\eta^2$	2
0	1	1	" "		

(d) parity yes, ang. mom no

(e)