PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #37

October 4, 1980

General Instructions

This Comprehensive Examination for Fall 1980 (#37) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you *may* find useful. Please return it at the end of the exam.

(20) 1. A plane polarized light wave in free space (ε=1, μ=1) falls normally on a semi-infinite dielectric having μ=1, ε=4. Calculate, from the conditions imposed by Maxwell's equations and the boundary conditions, what fraction of the wave's energy is transmitted into the dielectric, and what fraction is reflected.

2. You have a quantum mechanical system governed by a Hamiltonian H which is invariant under a unitary transformation U:

$$UHU^{-1} = H \qquad \text{or} \qquad UH = HU.$$

Then the eigenfunctions of H <u>can</u> be chosen to be simultaneous eigenfunctions of U.

(3) (a) Show that for nondegenerate states, eigenfunctions of H <u>must</u> be eigenfunctions of U.

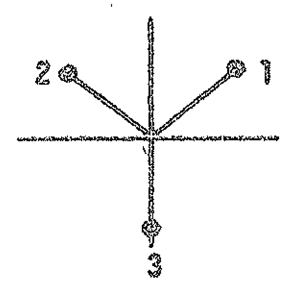
(3) (b) Show that for eigenstates ϕ_{λ_1}, ϕ_{λ_2} of U having eigenvalues $\lambda_1 \neq \lambda_2$ (even if they are not eigenfunctions of H),

$$\int \phi^*_{\lambda_2} H \phi_{\lambda_1} \, d\tau = 0.$$

(3) (c) As an example consider the motion in two dimensions of a particle in a field of equilateral triangular symmetry; that is, $U = R(2\pi/3)$ (rotation through 120°). Find the eigenvalues of U. How many are there?

(3) (d) Now assume that the potential has a strong cylindrically symmetric term plus a small perturbation with the symmetry of part (c). Show that the solutions of the unperturbed Hamiltonian are eigenfunctions of U.

(8) (e) Now consider the opposite extreme of a particle in the field of three identical, strong, short-range force centers located symmetrically at 30°, 150°, 270°. The ground state eigenfunctions of the same particle in the field of center 1, 2, or 3 alone are respectively ψ_1, ψ_2, ψ_3, each with energy E_0. Make up a trial function for approximate eigenstates of the triple potential with energies $E \approx E_0$. Determine these approximate eigenstates (within an overall constant) by the symmetry conditions alone and show that there are three different sets corresponding to the three different eigenvalues of U.



3. Consider two identifiable or classical atoms A and B, each of which can exist in one of two energy states $\varepsilon=0$ and $\varepsilon=\Delta$.

(6) (a) Determine the partition function Z for this system.

(14) (b) Utilize your result in (a), to determine an expression for the entropy S of the system and its value at high temperatures (i.e. $kT \gg \Delta$).

(8) 4. (a) Derive the equation for the final velocity of a rocket that <u>without</u> fuel has mass m_r and mass $m_r + M_f$ when full of fuel in a force-free region with a fuel-to-rocket relative velocity of u = const > 0 starting from $v_0 = 0$.

(6) (b) Now consider a two-stage rocket of initial mass $m_0 = m_1 + m_2 + M_1 + M_2$ of which the first fuel holder m_1 is dropped off after exhaustion of the first fuel M_1. The second fuel M_2 is then ignited and the rocket accelerates further. What is the final velocity?

(6) (c) For a given amount of fuel and a given payload, the two-stage rocket is advantageous over the one-stage because the dropped-off fuel holder m_1 doesn't have to be accelerated by M_2. Likewise, three stages are better than two, etc. Consider now the extreme case of a rocket with many small (infinitesimal) fuel holders which drop off as soon as their fuel is exhausted. The total mass is $m_p + m_h + M_f$, where

m_p = mass of payload
m_h = mass of holders
M_f = mass of fuel
$dM_f = \lambda dm_h$ where λ = constant.

(In other words each small holder holds a mass of fuel proportional to its own mass.) Calculate the final velocity of the payload after all fuel has been exhausted and all holders dropped off. Again assume constant relative velocity u of fuel to rocket.

5. Considering the electron to be a uniformly charged shell of radius r_0, calculate its mass in the following ways:

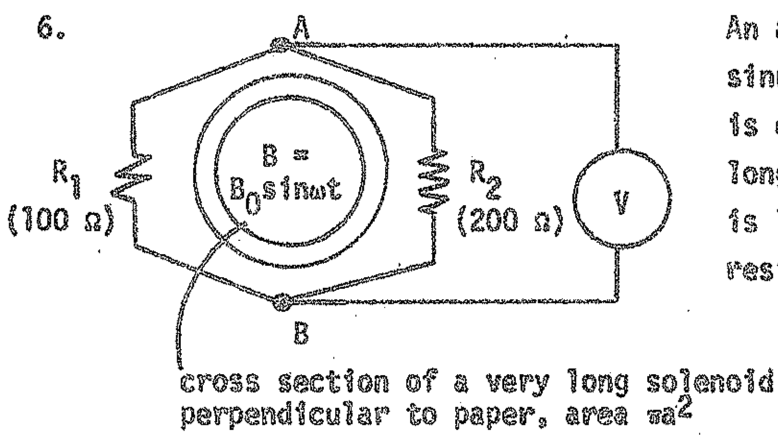
(5) (a) Consider the electron in uniform motion with velocity $u\hat{k}$, where $u \ll c$. Let the speed change slightly (i.e., neglect radiative effects); from the necessary reaction force which results from this change in motion, calculate the electron's mass.

(5) (b) For the electron at rest, calculate the electrostatic field energy W_0, and determine $m = \frac{W_0}{c^2}$.

(5) (c) For the moving electron, calculate the field momentum, and determine the associated mass.

(5) (d) Comment on the similarities and differences in the results of these calculations.

6.



An alternating current produces a sinusoidal B-field, $B = B_0 \sin\omega t$, which is everywhere longitudinal in a very long solenoid of radius a. The solenoid is linked by a circuit containing two resistances R_1 and R_2, as shown.

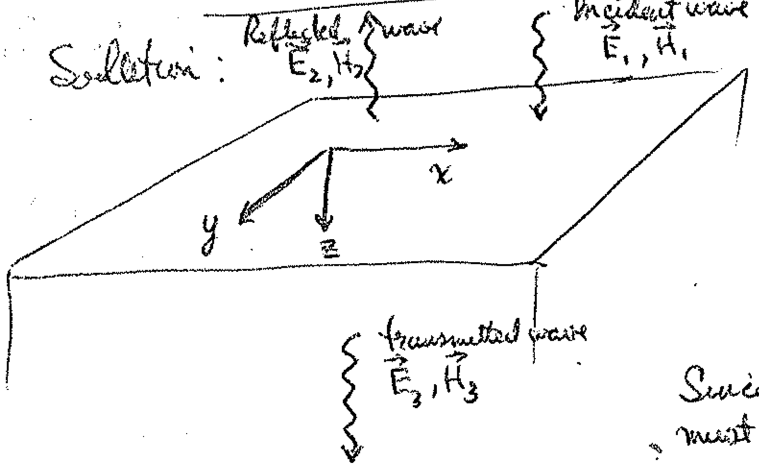
(7) (a) Determine the voltage drop across R_1 as a function of the time.

(3) (b) Determine the voltage drop across R_2 as a function of the time.

(7) (c) What will a high impedance AC voltmeter (draws no current) read when it is connected as shown, between A and B?

(3) (d) If R_1 and R_2 are interchanged, what will the meter read?

Solution:



1

The travelling waves in each region are, if we take the $\vec{E}$-polarization direction as x,

$$(E_x)_{outside} = E_{01} e^{i(k_1 z - \omega t)} + E_{02} e^{i(-kz - \omega t)}$$

$$(E_x)_{inside} = E_{03} e^{i(k_3 z - \omega t)}$$

Since the tangential component of $\vec{E}$ at the boundary $z=0$ must be continuous,

$$\underline{E_{01} + E_{02} = E_{03}} \quad ①$$

The $\vec{H}$ field is $\perp \hat{z}$ and to $\vec{E}$, such that

$$(H_y)_{outside} = H_{01} e^{i(k_1 z - \omega t)} + H_{02} e^{i(-kz - \omega t)}$$

$$(H_y)_{inside} = H_{03} e^{i(k_3 z - \omega t)}$$

and, at the boundary, the tangential component of $\vec{H}$ is also continuous

$$\underline{H_{01} + H_{02} = H_{03}} \quad ②$$

Maxwell's equations require that $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ so for terms like $E_x = E_0 e^{i(kz - \omega t)}$

$$\frac{\partial B_y}{\partial t} = -\left(\frac{\partial E_x}{\partial z} - \underbrace{\frac{\partial E_z}{\partial x}}_{=0}\right) = -ikE_x$$

which we can integrate over time to get $B_y = \frac{-ikE_x}{-i\omega} = \frac{k}{\omega} E_x = \frac{1}{v} E_x$ since $\frac{\omega}{k} = v = \frac{1}{\sqrt{\mu \varepsilon}}$ from the wave equation

Then $H_y = \frac{B_y}{\mu} = \frac{E_x}{\mu v} = \sqrt{\frac{\varepsilon}{\mu}} E_x$ or $E_x = \sqrt{\frac{\mu}{\varepsilon}} H_y$

$$\text{Thus,} \quad \left.\begin{aligned} E_{01} &= \sqrt{\frac{\mu_1}{\varepsilon_1}} H_{01} = H_{01} \\ E_{02} &= -\sqrt{\frac{\mu_1}{\varepsilon_1}} H_{02} = -H_{02} \\ E_{03} &= \sqrt{\frac{\mu_3}{\varepsilon_3}} H_{03} = \sqrt{\frac{1}{4}} H_{03} = \frac{1}{2} H_{03} \end{aligned}\right\} ③$$

The $-$ sign in E_{02} is there because the reflected wave must propagate in the $-z$ direction.

Using ③, ② becomes $E_{01} - E_{02} = 2E_{03}$

and ① is $E_{01} + E_{02} = E_{03}$

adding $2E_{01} = 3E_{03}$ while subtracting, $2E_{02} = -E_{03} = -\frac{2}{3}E_{01}$

$\underline{E_{03} = \frac{2}{3}E_{01}}$ (4a) $\underline{E_{02} = -\frac{1}{3}E_{01}}$ (4b)

Similarly, using ③, ① becomes

$H_{01} - H_{02} = \frac{1}{2}H_{03}$

and ② is $H_{01} + H_{02} = H_{03}$

adding, $2H_{01} = \frac{3}{2}H_{03}$ while if we subtract, we get $2H_{02} = \frac{1}{2}H_{03} = \frac{1}{2}\cdot\frac{4}{3}H_{01}$

$\underline{H_{03} = \frac{4}{3}H_{01}}$ (4c) $\underline{H_{02} = \frac{1}{3}H_{01}}$ (4d)

The Poynting Vector $\left(\frac{\text{watts}}{m^2}\right)$ at $z=0$ for the incident wave is the real part of $\vec{E}_1 \times \vec{H}_1$,

$P_1 = (E_{x1} H_{y1})_{\substack{real \\ z=0}} = E_{01} H_{01} \cos^2 \omega t$ whose average value is $\frac{1}{2}E_{01}H_{01}$

so $\underline{\bar{P}_1 = \frac{1}{2}E_{01}^2}$ using ③

Similarly, for the reflected wave, $\bar{P}_2 = \frac{1}{2}E_{02}H_{02} = \frac{1}{2}\left(-\frac{1}{3}E_{01}\right)\left(-\frac{1}{3}E_{01}\right) = \frac{1}{9}\left(\frac{1}{2}E_{01}^2\right) = \underline{\underline{\frac{1}{9}\bar{P}_1}}$

and for the transmitted wave, $\bar{P}_3 = \frac{1}{2}E_{03}H_{03} = \frac{1}{2}\left(\frac{2}{3}E_{01}\right)\left(\frac{4}{3}E_{01}\right) = \underline{\underline{\frac{8}{9}\bar{P}_1}}$

So $\bar{P}_2 + \bar{P}_3 = \bar{P}_1$ as required by energy conservation,

and $\frac{1}{9}$ is reflected, $\frac{8}{9}$ transmitted.

VM

(2)

QM solns.

(a) $$H\psi = E\psi$$
$$UH\psi = HU\psi = EU\psi$$
$$\Rightarrow U\psi = \lambda\psi \quad \text{for } \psi \text{ nondegen.}$$

(b) $$\int \psi_{\lambda_2}^* H \psi_{\lambda_1} \, d\tau = \frac{\int \psi_{\lambda_2}^* [U, H] \psi_{\lambda_1} \, d\tau}{\lambda_2 - \lambda_1} = 0$$

(c) $$U\psi_\lambda = \lambda \psi_\lambda$$
$$U^3 \psi_\lambda \equiv \psi_\lambda = \lambda^3 \psi_\lambda$$
$$\lambda^3 = 1$$

$$(\lambda^3 - 1) \quad 0 \quad (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$
$$\lambda = 1, \ \lambda = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1-4}$$
$$= \tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i$$

(d) Solutions in central field are $\psi_m = \varphi(r) e^{im\varphi}$
$$U\psi_m = \varphi(r) U e^{im\varphi} = \varphi(r) e^{im(\varphi - 2\pi/3)}$$
$$= e^{-2\pi i m/3} \varphi(r) e^{im\varphi}$$

ψ_m are eigenstates of U.

(e) Trial fn $\quad \psi = a_1\psi_1 + a_2\psi_2 + a_3\psi_3$
$$U\psi = a_1 U\psi_1 + a_2 U\psi_2 + a_3 U\psi_2$$
$$= a_1\psi_2 + a_2\psi_3 + a_3\psi_1 = \lambda\psi$$
$$U^2\psi = a_1\psi_3 + a_2\psi_1 + a_3\psi_2 = \lambda^2\psi$$

comparing we get $\quad a_3 = \lambda a_1$
$$a_2 = \lambda a_3 = \lambda^2 a_1$$

thus $\quad \psi = a_1(\psi_1 + \lambda^2\psi_2 + \lambda\psi_3)$ for the 3 values of λ given in part c. $\Rightarrow$ 3 solns.

λ

3

(i) $Z = \sum_i e^{-\beta \varepsilon_i} = 1 + 2e^{-\Delta/kT} + e^{-2\Delta/kT}$

(ii) $S = \dfrac{\langle E \rangle}{T} + k \ln Z$, from $F = \langle E \rangle - TS$
and $F = -kT \ln Z$.

But $\langle E \rangle = -\dfrac{\partial}{\partial \beta} \ln Z = kT^2 \dfrac{\partial}{\partial T} \ln Z = \dfrac{kT^2}{Z} \dfrac{\partial Z}{\partial T}$

$\therefore\ S = \dfrac{kT}{Z}\dfrac{\partial Z}{\partial T} + k \ln Z$ (general expression)

as $kT \gg \Delta$ we get

$$S = \frac{kT}{1 + 2(1 - \Delta/kT) + (1 - 2\Delta/kT)} \left\{ \frac{2\Delta}{kT^2}\left(1 - \frac{\Delta}{kT}\right) + \frac{2\Delta}{kT^2}\left(1 - \frac{2\Delta}{kT}\right) \right\}$$

$$+ k \ln \left\{ 1 + 2(1 - \Delta/kT) + (1 - 2\Delta/kT) \right\}$$

$$= \frac{\Delta}{T} + k \left\{ \ln 4 + \ln(1 - \Delta/kT) \right\}$$

and since $\ln(1+x) \approx x$ for $x \ll 1$, we get

$$\underset{(kT \gg \Delta)}{S} \cong \frac{\Delta}{T} + k \ln 4 - \frac{k\Delta}{kT} = k \ln 4 .$$

(4) Dynamics solns

(a) momentum at t $= m v$

momentum at $t+dt$

$$(m+dm)(v+dv) + (v-u)(-dm)$$

momentum conservation

$$(m+dm)(v+dv) - (v-u)\,dm = mv$$

$$\cancel{mv} + \cancel{dm\,v} + m\,dv - \cancel{v\,dm} + u\,dm = \cancel{mv}$$

$$u\,dm = -m\,dv$$

$$dv = -u\,\frac{dm}{m}$$

$$v = \text{const} - u \ln m \qquad (1)$$

In general $v = v_0$ at $m = m_r + M_f$

$$\text{const} = v_0 + u \ln (m_r + M_f)$$

$$v = v_0 + u \ln \frac{m_r + M_f}{m}$$

the end velocity is at $m = m_r$

$$v_{end} = v_0 + u \ln \left(1 + \frac{M_f}{m_r}\right)$$

In our case $v_{end} = u \ln \left(1 + \frac{M_f}{m_r}\right)$

(b) In first stage at $v=0$ $m = m_1 + m_2 + M_1 + M_2$

$$\text{const} = u \ln (m_1 + m_2 + M_1 + M_2)$$

$$v_f = u \ln \frac{m_1 + m_2 + M_1 + M_2}{m_1 + m_2 + M_2}$$

(b) (continued) For the second stage the initial and final mass are $m_2 + M_2$ and m_2 respectively so

$$v_f = u \ln \frac{m_1 + m_2 + M_1 + M_2}{m_1 + m_2 + M_2} + u \ln \frac{m_2 + M_2}{m_2}$$

$$= u \ln \left(1 + \frac{M_1}{m_1 + m_2 + M_2}\right)\left(1 + \frac{M_2}{m_2}\right)$$

(c)

momentum at t : mv

momentum at $t+dt$: $(m+dm)(v+dv) - v\,dm_h - (v-u)\,dM_f$

$$\cancel{mv} = \cancel{mv} + \cancel{v\,dm} + m\,dv - \cancel{v\,dm_h} - \cancel{v\,dM_f} + u\,dM_f$$

$$dv = -u\,\frac{dM_f}{m}$$

$$dM_f + dm_h = \left(1 + \frac{1}{\rho}\right) dM_f = \frac{\rho+1}{\rho}\,dM_f$$

$$dv = -u\,\frac{\rho}{\rho+1}\,\frac{dm}{m}$$

$$v = -u \frac{\rho}{1+\rho} \ln m + \text{const}$$

$v(0) = 0, \quad m(0) = m_p + m_h + m_f$

at end of fuel $\quad m = m_p$

$$\text{const} = u \frac{\rho}{1+\rho} \ln\left(m_p + m_h + M_f\right)$$

$$v_{end} = u \frac{\rho}{1+\rho} \ln\left(\frac{m_p + m_h + M_f}{m_p}\right)$$

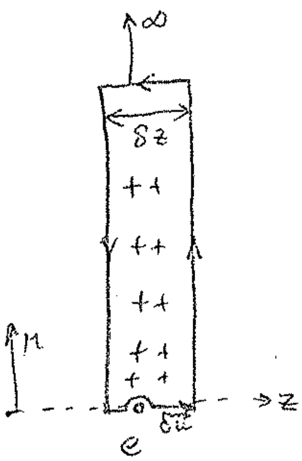
$$= u \frac{\rho}{1+\rho} \ln\left(1 + \frac{m_h + M_f}{m_p}\right)$$

Note that this can become very large for a very small payload.

(5)

a) If the speed is changed by a small amount δu, the vector potential changes by
$$\delta\vec{A} = \frac{\mu_0 e}{4\pi r}\,\delta\vec{u}$$

The magnetic flux passing through an area in a narrow strip perpendicular to $\delta\vec{u}$ and extending from r_0 to ∞ will change like



$$\delta\Phi = \oint \delta\vec{A}\cdot d\vec{l}$$

and will produce an electric field at the electron which opposes the change in $\vec{u}$ (Lenz's law).

$$\vec{F} = e\vec{E} = -e\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 e^2}{4\pi r_0}\frac{d\vec{u}}{dt} = -m\frac{d\vec{u}}{dt}$$

$$\text{so } m = \frac{\mu_0 e^2}{4\pi r_0}\left(\frac{c^2}{c^2}\right) = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_0 c^2}$$

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b) $$W_0 = \frac{1}{2}\int \varepsilon_0 E^2\,dv = \frac{4\pi\varepsilon_0}{2}\int_{r_0}^{\infty}\left(\frac{e}{4\pi\varepsilon_0 r^2}\right)^2 r^2\,dr = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_0} = mc^2$$

$$\text{So } m = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_0 c^2}$$

c) The field carries momentum density $\vec{g} = \dfrac{\vec{E}\times\vec{H}}{c^2}$ and therefore

momentum $\quad \vec{G} = \displaystyle\int \frac{\vec{E}\times\vec{H}}{c^2}\, dv$

For quasi-static fields, $\vec{E}$ and $\vec{H}$ are related by $\vec{H} = \varepsilon_0(\vec{u}\times\vec{E})$

so $\vec{G} = \displaystyle\int \frac{\varepsilon_0 \vec{E}\times(\vec{u}\times\vec{E})}{c^2}\, dv = \frac{\varepsilon_0}{c^2}\int \vec{u}E^2 - (\vec{u}\cdot\vec{E})\vec{E}\; dv$

but $\vec{G}$ is axially symmetric about $\hat{k}$, and $\vec{G}$ is along $\vec{u}$, so the 2nd term is just $\vec{u}\,E_z^2$. Since for $u \ll c$ $\vec{E}$ is spherically symmetric, and

$$E^2 = E_x^2 + E_y^2 + E_z^2$$

$$\vec{G} = \frac{\varepsilon_0}{c^2}\int_{r_0}^{\infty}\left(E_x^2 + E_y^2 + \cancel{E_z^2} - \cancel{E_z^2}\right)\vec{u}\, dv = \frac{\varepsilon_0}{c^2}\int_{r_0}^{\infty}\left(\tfrac{2}{3}E^2\right)\vec{u}\, dv = \frac{4}{3}\frac{\vec{u}}{c^2}\left[\frac{1}{2}\int_{r_0}^{\infty}\varepsilon_0 E^2\, dv\right]$$

$$= \left(\frac{4}{3}\frac{W_0}{c^2}\right)\vec{u}$$

$$= m\,\vec{u}$$

So $\underline{\underline{m = \dfrac{4}{3}\dfrac{W_0}{c^2}}}$

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d) The mass in part a) is the "inertial mass", of electromagnetic origin and is an effective mass which results from the inertial reaction of the magnetic field on the electron. It turns out to be the same as in part b), but these are only estimates, because r_0 is model dependent.

In part c), the factor r_0 doesn't appear explicitly, so the result is not model dependent (except that the integral for G would diverge if the electron were a point charge). However, an additional mass $\dfrac{W_0}{3c^2}$ appears, while the "correct" mass must be given by $\dfrac{W_0}{c^2}$ if relativity is correct. This extra energy (or mass) represents non-electromagnetic binding; Earnshaw's theorem says that the electron's charge system by itself would be unstable otherwise.

Soln: 6

a) Faraday's Law tells us that $V = -\frac{d\Phi}{dt}$ where $\Phi = \int \vec{B} \cdot d\vec{s} = \int B\,ds = B_0 \sin\omega t (\pi a^2)$

So in the loop AR_1BR_2A the induced voltage is

$$V = -\frac{d\Phi}{dt} = -\omega B_0 \pi a^2 \cos\omega t$$

and the current flowing is $I = \frac{V}{R_{total}} = -\frac{\omega B_0 \pi a^2 \cos\omega t}{R_1 + R_2} = -\frac{\omega B_0 \pi a^2 \cos\omega t}{300}$

So $V_1 = IR_1 = -\frac{\omega B_0 \pi a^2 \cos\omega t}{300}(100) = \underline{\underline{-\frac{1}{3}\omega B_0 \pi a^2 \cos\omega t}}$

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b) $V_2 = IR_2 = -\frac{\omega B_0 \pi a^2 \cos\omega t}{300}(200) = \underline{\underline{-\frac{2}{3}\omega B_0 \pi a^2 \cos\omega t}}$

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c) To get the voltmeter reading, we must include the meter in a path with no induced emf; this path is $B\,Ⓥ\,A R_2 B$, which includes no flux change. The voltmeter reads only the RMS value of the voltage drop across R_2;

$$V_{voltmeter} = (V_2)_{RMS} = \underline{\underline{\frac{\sqrt{2}}{2}\left(\frac{2}{3}\omega B_0 \pi a^2\right)}}$$

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d) If R_1 and R_2 are interchanged, then

$$V_{voltmeter} = (V_1)_{RMS} = \underline{\underline{\frac{\sqrt{2}}{2}\left(\frac{1}{3}\omega B_0 \pi a^2\right)}},$$ half the previous reading.