PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #36

March 29, 1980

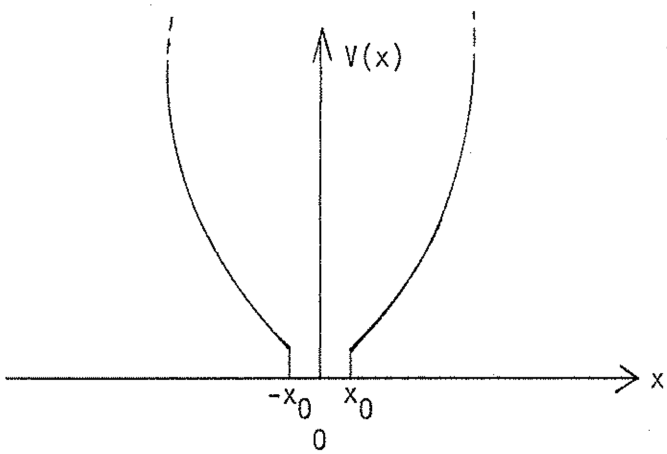
General Instructions

This Comprehensive Examination for Spring 1980 (#36) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

1. Given, a one-dimensional simple harmonic oscillator in which only the first three eigenstates need be considered. A perturbation then acts such that the total potential energy curve has the following form



$V(x) = 0; \quad -x_0 \le x \le x_0$

$V(x) = \frac{1}{2} Cx^2; \begin{cases} x < -x_0 \\ x > +x_0 \end{cases}$

where $\left| \frac{1}{2} Cx_0^{\,2} \right| \ll E_0$, E_1 and E_2 (the unperturbed energies),

and where $x_0 \ll a$ (the length parameter).

(10) (i) Determine (up to 2nd order) the shift in the ground state energy.

(10) (ii) Calculate the lowest order correction to the ground state wave function.

2. Consider a two-dimensional monatomic gas, i.e., one whose molecules can move freely in a plane but are confined within an area A. Assume that the molecules obey Boltzmann statistics and that the molecules are point particles which exert no force on one another except when they collide.

(5) (a) Find the partition function.

(5) (b) Find the velocity distribution function.

(5) (c) Use the results from (b) to calculate the force per unit length (i.e., the "pressure") which the gas exerts on its two-dimensional container. Express this pressure in terms of the temperature to get the equation of state.

(5) (d) Calculate the molar specific heat of the gas at constant volume.

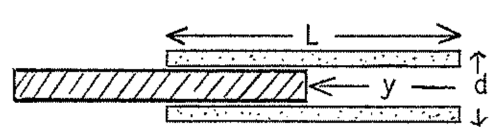
Do not guess the results from the three-dimensional case--you must work them out from first principles.

3. A parallel plate capacitor has the region between its plates completely filled with a dielectric slab of dielectric constant "K" and density "ρ". The plates are separated by a distance "d" and each has width "w" and length "L". The capacitor is initially connected to a battery of voltage "U_0".

(6) (a) If the battery is disconnected and the dielectric slab is partially withdrawn a distance "y" along the L dimension (see diagram), what is the potential difference between the plates?

(7) (b) What is the magnitude and direction of the force on the dielectric when it is displaced by this length y? (You may ignore fringing effects.)

(7) (c) If the dielectric is released from rest after being withdrawn a distance $y \ll L$, describe its ensuing motion. In particular, if the motion is periodic, what is the period?



4. Given the one-dimensional potential:

$$V(x) = \begin{cases} \dfrac{\hbar^2}{2m}\left[\, \alpha^2 x^2 + \dfrac{\beta(\beta+1)}{x^2} \,\right] & x > 0 \\[2ex] \infty & x < 0 \end{cases} ,$$

one would expect solutions to the Schroedinger equation which are similar to the ordinary harmonic oscillator.

(15) (a) Show* that the energy levels for this potential are given by

$$E_n = \frac{\alpha^2\hbar^2}{2m}\left[\, 4n + 2\beta + 3 \,\right]$$

for β real and positive. (*Explain the reason for any assumptions made in deriving this result.)

(2) (b) By making an appropriate redefinition of α, compare the answer of part (a) to the ordinary harmonic oscillator with regard to level spacing and ground-state energies.

(3) (c) For $\beta = 0$ this potential is the half-harmonic oscillator. Compare the $\beta = 0$ levels with the usual harmonic oscillator and explain why certain states are "missing".

(20) 5. In this problem you are asked to derive the equation of state [i.e., $pV = f(T)$] for a system of N non-interacting bosons. You are given the information that the grand partition function

$Q = \sum_N z^N \mathcal{Z}_N$, where $\mathcal{Z}_N$ is the partition function,

and that $N = \left(\dfrac{\partial \log Q}{\partial \log z} \right)_{\beta,V} \equiv z\left(\dfrac{\partial \log Q}{\partial z} \right)_{\beta,V}$.

For non-interacting bosons it is also true that

$\dfrac{pV}{kT} \equiv \log Q = -\sum_i \log\left[\, 1 - z \exp(-\beta\varepsilon_i) \,\right]$, where i labels the single particle states.

HINTS: (1) The levels can be assumed to be densely packed.

(2) Integrands should be expanded and then integrated.

6. A spherical electrode of radius a is surrounded by a concentric spherical electrode of radius b, while the intervening space is filled with a medium whose conductivity is inversely proportional to the distance from the center of the system, i.e., $\sigma = K/r$. The outer sphere is maintained at potential ϕ_0, and the total current I flows between the electrodes.

(6) (a) Use conservation of current to find the permittivity of the conducting medium as a function of r.

(8) (b) Find the potential at a distance r ($b > r > a$) from the center. Express your result in terms of K and I.

(6) (c) Find the bound polarization charge density in the medium.

Solutions to Comp. Exam # 36

$\boxed{\#1}$

$$E_0' = E_0 + \langle 0|v|0\rangle - \sum_{\substack{m \\ (m\neq 0)}} \frac{|\langle m|v|0\rangle|^2}{E_m - E_0}$$

where $E_m = (m+\frac{1}{2})\hbar\omega$ and
$$v = \begin{cases} -\frac{Cx^2}{2} & ; \quad -x_0 \leq x \leq +x_0 \\ 0 & ; \quad x < -x_0,\ x > +x_0 . \end{cases}$$

Also, $\psi_0' = a_{00}\psi_0 - \sum_{\substack{m \\ (m\neq 0)}} \frac{\langle m|v|0\rangle}{E_m - E_0}\psi_m .$

where
$$\langle m|v|0\rangle \equiv \int_{-\infty}^{\infty} \psi_m^*(x)\, v(x)\, \psi_0(x)\, dx .$$

EXAMINE THE <u>THREE</u> <u>POSSIBLE</u> <u>TERMS</u>:

① $\langle 0|v|0\rangle = -\frac{C}{2a\sqrt{\pi}}\int_{-x_0}^{+x_0} x^2 e^{-x^2/a^2}dx \approx -\frac{Cx_0^3}{3a\sqrt{\pi}}\left(1 - \frac{3x_0^2}{5a^2}\right)$

where $e^{-x^2/a^2} \simeq 1 - \frac{x^2}{a^2}$ over range of integration.

② $\langle 1|v|0\rangle = 0$, due to odd function integrand.

and

③ $\langle 2|v|0\rangle \simeq +\frac{C}{4\sqrt{2\pi}\,a}\left(\frac{4}{3}x_0^3 - \frac{12}{5}\frac{x_0^5}{a^2}\right)$

$$\therefore \quad E_0' = E_0 - \frac{Cx_0^3}{3\sqrt{\pi}\,a} + \frac{Cx_0^5}{5\sqrt{\pi}\,a^3} - \frac{C^2x_0^6}{36\pi a^2\hbar\omega}$$

and $\psi_0' = a_{00}\psi - \frac{\langle 2|v|0\rangle}{E_2 - E_0}\psi_2 = a_{00}\psi_0 - \frac{Cx_0^3}{6\sqrt{2\pi}\,a\hbar\omega}\psi_2$

where a_{00} can be determined via normalization condition : $\quad a_{00}^2 + \left(\frac{Cx_0^3}{6\sqrt{2\pi}\,a\hbar\omega}\right)^2 = 1 .$

#2

Going to integral, $\frac{pV}{kT} = -\int_0^\infty \log\left(1 - z e^{-\beta p^2/2m}\right) \frac{V}{(2\pi\hbar)^3} 4\pi p^2 dp$ — ①

$$N = z\frac{\partial}{\partial z}\left(\frac{pV}{kT}\right)_{\beta, V} = \frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty \frac{z\, e^{-\beta p^2/2m}}{(1 - z e^{-\beta p^2/2m})} p^2 dp \quad \text{———} ②$$

Note that we lose V from both sides of ① and so in order to put together our equation of state we need to evaluate ② as well.

Starting on ②; let $x^2 = \beta p^2/2m$, then we get

$$\frac{N}{V} = \frac{4\pi}{(2\pi\hbar)^3}\left(\frac{2m}{\beta}\right)^{3/2}\int_0^\infty z x^2 e^{-x^2}(1 - z e^{-x^2})^{-1} dx$$

$$= \cdots\cdots \left\{ z\int_0^\infty x^2 e^{-x^2} dx + z^2\int_0^\infty x^2 e^{-2x^2} dx + \cdots \right\}$$

having expanded the term in parentheses.

Let's examine the form of the integral(s) above, i.e.

$\int_0^\infty x^2 e^{-\ell x^2} dx$. Put $u = \ell x^2$ then this integral becomes

$$\frac{1}{2\ell^{3/2}}\int_0^\infty u^{1/2} e^{-u} du = \frac{\Gamma(3/2)}{2\ell^{3/2}} = \frac{\sqrt{\pi}}{4\ell^{3/2}}$$

and so the complete expression in the brackets becomes $\sum_{\ell=1}^\infty \frac{z^\ell}{\ell^{3/2}} \cdot \frac{\sqrt{\pi}}{4}$

$$\therefore \quad \frac{N}{V} = \frac{1}{(2\pi\hbar)^3}\left(\frac{2m\pi}{\beta}\right)^{3/2}\sum_{\ell=1}^\infty \frac{z^\ell}{\ell^{3/2}} \quad \text{———} ③$$

To get ① evaluated we need to expand $\log(1-\xi)$

$$\log(1-\xi) = -\xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} - \cdots$$ and again

make use of $x^2 = \frac{\beta p^2}{2m}$ as used earlier, obtaining

$$\frac{p}{kT} = \frac{4\pi}{(2\pi\hbar)^3}\left(\frac{2m}{\beta}\right)^{3/2} \int_0^\infty \left[z x^2 e^{-x^2} + \frac{z^2}{2} x^2 e^{-2x^2} + \frac{z^3}{3} x^2 e^{-3x^2} + \cdots \right] dx$$

You can save time by recognizing that this expression is similar to earlier expression for $\frac{N}{V}$. This one just has an extra l in the denominator of each term in the sum.

$\therefore$

$$\frac{p}{kT} = \frac{1}{(2\pi\hbar)^3}\left(\frac{2m\pi}{\beta}\right)^{3/2} \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} \quad \text{———} \quad ④$$

Taking the ratio of ④/③ one gets

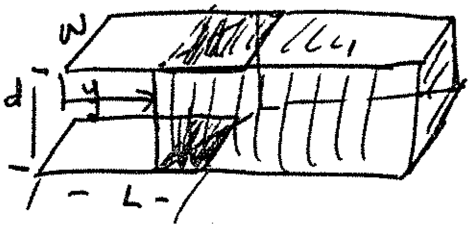
$$\frac{pV}{NkT} = \frac{\sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}}}{\sum_{l=1}^{\infty} \frac{z^l}{l^{3/2}}}.$$

$\therefore$

$$pV = NkT \left\{ \frac{z + z^2/2^{5/2} + z^3/3^{5/2} + z^4/4^{5/2} + \cdots}{z + z^2/2^{3/2} + z^3/3^{3/2} + z^4/4^{3/2} + \cdots} \right\}$$

Solution #3 EM-Undergrad

(6) (a) Charge is conserved, yet potential difference changes:



$Q = C_0 U_0 = \left(\frac{K\varepsilon_0 LW}{d}\right) U_0 = \text{fixed}.$

This, is like 2 parallel capacitors.

$$Q_{tot} = Q_1 + Q_2$$

$$\frac{K\epsilon_0 LW}{d} U_0 = \frac{\varepsilon_0 W(y)U}{d} + \frac{K\varepsilon_0 W(L-y)U}{d}$$

$$\boxed{U = \frac{KLU_0}{LK - y(K-1)}} \quad \text{for } 0 \leq y \leq L$$

$$\boxed{U = KU^0 \quad \text{for } y \geq L}$$

(7) (b) $\text{Force} = -\partial W/\partial y\Big)_Q$

$$W = \tfrac{1}{2} C U^2 = \tfrac{1}{2} C_{equiv} U^2$$

$$C_{equiv} = C_1 + C_2 = \epsilon_0 W y/d + \frac{\varepsilon_0 K(L-y)W}{d} = \frac{\varepsilon_0 W}{d}[LK - y(K-1)]$$

$$W = \frac{1}{2}\frac{\varepsilon_0 W}{d}[LK - y(K-1)]\frac{K^2L^2U_0^2}{[LK-y(K-1)]^2}$$

$$= \frac{1}{2}\frac{\varepsilon_0 W}{d}\frac{K^2L^2U_0^2}{[LK-y(K-1)]}$$

$$F = -\frac{\partial W}{\partial y}\Big)_Q = -\frac{1}{2}\frac{\varepsilon_0 W K^2L^2U_0^2}{d}\frac{-1(-(K-1))}{[LK-y(K-1)]^2}$$

$$F = -\frac{\varepsilon_0 W K^2L^2U_0^2(K-1)}{2d[LK-y(K-1)]^2}$$

pulling slab in (decrease y) for $0 < y < L$

$F = 0$ for $y \geq L$ or $y = 0$!!

no fringing field

(c) The force is essentially constant for very small
(7) y, not S.H.M. (linear)

$$F \simeq \frac{-\varepsilon_0 w U_0^2 (K-1)}{2d} \qquad \text{as long as } y \neq 0, L$$

so slab is pulled back - then keeps going on the other side, where it extends ~~...~~, stops (E is conserved) and then repeats — NOT S. H. M, get periodic.

For constant acceleration,

$$y = \tfrac{1}{2} a t^2$$

$$t = \sqrt{\frac{2y}{a}} \quad = \text{time for } 1/4 \text{ of a period}$$

$$T = 4\sqrt{2y/a}$$

$$a = \text{Force / mass} = \frac{+\varepsilon_0 \cancel{w} U_0^2 (K-1)}{2d} \Big/ \rho \cancel{w} L d = \frac{\varepsilon_0 U_0^2 (K-1)}{2\rho L d^2}$$

$$\boxed{T = 4\sqrt{\frac{4 y \rho L d^2}{\varepsilon_0 U_0^2 (K-1)}}}$$

Solution : Grad. Q.M. #4

(a) Schroedinger Eqtn: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$

$+\frac{\hbar^2}{2m}[\alpha^2x^2+\beta(\beta+1)/x^2]\psi(x) = E\psi(x)$

let $\varepsilon = \frac{2mE}{\alpha^2\hbar^2}$, $\xi = \sqrt{\alpha}\, x$ (for Dimensionless form), then

$$\frac{d^2\psi}{d\xi^2} - \left(\xi^2 - \varepsilon + \frac{\beta(\beta+1)}{\xi^2}\right)\psi(\xi) = 0$$

for $\xi \to \infty$: $\frac{d^2\psi}{d\xi^2} \cong \xi^2\psi(\xi)$

$\psi(\xi) \underset{\xi\to\infty}{\sim} e^{-\xi^2/2}$ (want to remove this)

$\therefore$ assume

$$\psi(\xi) = e^{-\xi^2/2}u(\xi)$$

$$\Rightarrow \quad \frac{d^2u}{d\xi^2} - 2\xi\frac{du}{d\xi} - \left(1-\varepsilon+\frac{\beta(\beta+1)}{\xi^2}\right)u(\xi) = 0$$

assume a power series for $u(\xi)$ - but from form of eqtn it is evident that only terms in ξ^2 need be considered.

$$u(\xi) = \xi^\gamma \sum_{n=0}^{\infty} a_n \xi^{2n}$$: where γ can be even-odd etc

$$\Rightarrow \sum_{n=-1}^{\infty} a_{n+1}\{(\gamma+2n+2)(\gamma+2n+1)-\beta(\beta+1)\}\xi^{2n} = \sum_{n=0}^{\infty} a_n\{2(\gamma+2n)+(1-\varepsilon)\}\xi^{2n}$$

equate coefficients of each power to zero gives for

$n=-1$: $a_0[\gamma(\gamma-1)-\beta(\beta+1)] = 0$

$n \geq 0$ $\quad a_{n+1} = a_n \dfrac{1+2\gamma+4n-\varepsilon}{(\gamma+2n+2)(\gamma+2n+1)-\beta(\beta+1)}$ $\quad n = 0,1,2,\ldots\infty$

$a_0 \neq 0$ (ξ^γ removes any power) $\Rightarrow \gamma(\gamma-1) = \beta(\beta+1)$

$\Rightarrow \gamma = 1+\beta, -\beta$

Since $\boxed{\psi(0)=0}$ and $\beta \geq 0$ $\Big\}$ $\boxed{\gamma = 1+\beta}$

thus $\quad a_{n+1} = a_n \dfrac{3+2\beta+4n-\varepsilon}{2(1+n)(3+2\beta+2n)}$.

but, for very large n, $a_{n+1} \simeq 1/n\, a_n$ and so unless the series terminates, $u(\xi) \sim \exp(\xi^2)$ & $\psi(\xi)$ diverges (i.e. can't be normalized) $\therefore$ must terminate series $\Rightarrow$

$$3+2\beta+4n-\varepsilon = 0$$

$$\varepsilon = 3+2\beta+4n = \frac{2mE}{\hbar^2\alpha^2}$$

$$\boxed{E_n = \frac{\alpha^2\hbar^2}{2m}\left[3+2\beta+4n\right]} \qquad n = 0, 1, 2, \ldots \infty$$

(b) If define α: $\dfrac{\hbar^2\alpha^2}{2m} = \frac{1}{2}m\omega^2$

$$\boxed{E_n = \hbar\omega\left[2n+\beta+3/2\right]}$$

So the g.s. energy is $E_0 = \hbar\omega(\beta+3/2)$
and the spacing is $2\hbar\omega$

The ordinary harmonic oscillator has energies

$$E_{n'}' = \hbar\omega(n'+1/2) \qquad n' = 0, 1, 2 \ldots \infty$$

$E_0 = \hbar\omega/2$,
Spacing $\hbar\omega$

So, the levels in this problem is twice that of ordinary h.o. and the g.s. is approximately 3 times as that h.o. (small β)

(c) $\beta = 0 \Rightarrow$ $E_n = \hbar\omega\left[2n+3/2\right]$ "half harmonic oscillator"

$E_n = \hbar\omega\left[(2n+1)+1/2\right]$

i.e. $n' = 2n+1 = 1, 3, 5 \ldots$ only.

The even n' are missing because $V(x<0)=\infty$ forces $\psi(0)=0$ & $\therefore$ excludes symmetric (even) wave functions.

Stat. mech. – Solution

a) $$Z = \sum_i \exp(-m v_i^2 / 2kT)$$

where $v_i^2 = v_{ix}^2 + v_{iy}^2$

To pass from $\Sigma \Rightarrow \int$ multiply + divide by

$$H \equiv dx\, dy\, dv_x\, dv_y$$

$$Z = \frac{1}{H} \int dx\, dy\, dv_x\, dv_y\, e^{-mv^2/2kT}$$

$$= \frac{A}{H} \left(\frac{2\pi kT}{m}\right)$$

b) The number of particles in the i'th cell

$$N_i = \frac{N}{Z} e^{-w_i/kT} = \frac{NH}{A}\left(\frac{m}{2\pi kT}\right) e^{-mv^2/2kT}$$

i.e., $$d^4N = \frac{N}{A}\left(\frac{m}{2\pi kT}\right) e^{-mv^2/2kT}\, dx\, dy\, dv_x\, dv_y$$

Integrating out $dx\, dy$:

$$d^2N = N \left(\frac{m}{2\pi kT}\right) e^{-mv^2/2kT}\, dv_x\, dv_y$$

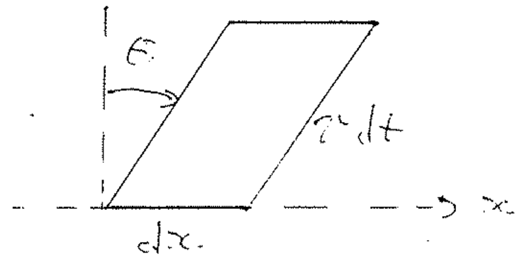
$$dn_v = \int \frac{d^2N}{dv_x\, dv_y} \frac{\partial(v_x, v_y)}{\partial(v, \theta)} \frac{d\theta\, dv}{N}$$

$$= \frac{m}{kT} v\, e^{-mv^2/2kT}$$

c) The fraction of molecules with velocities between v and $v + dv$ and angles between θ and $\theta + d\theta$ is

$$\frac{dn_v\, d\theta}{2\pi}$$

The number of these molecules in the parallelogram shown



is $\quad \dfrac{N}{A}\left(\dfrac{dn_v\, d\theta}{2\pi}\right)\left(v \cos\theta \, dx \, dt\right)$

Each molecule in the parallelogram collides at dx yielding a change in momentum $\Delta p = 2mv\cos\theta$

$$\text{"pressure"} = \left\langle \frac{dF}{dx} \right\rangle = \int_0^\infty \frac{d^3 p}{dt\, dx\, dv}\, dv$$

$$= \left(\frac{N}{2\pi}\right) \times \frac{2m}{A} \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta \int_0^\infty v^2 \, dn_v$$

$$= \frac{1}{2}\frac{N}{A} m \overline{v^2}$$

but $\overline{v^2} = \displaystyle\int_0^\infty v^2 dn_v = \int_0^\infty \frac{m}{kT} v^3 e^{-mv^2/2kT}\, dv$

$$= \frac{m}{kT} \cdot \frac{1}{2}\left(\frac{2kT}{m}\right)^2 = \frac{2kT}{m}$$

and $AP = NkT$

d) $\quad U = NkT^2 \dfrac{d(\ln Z)}{dT} = NkT^2 \dfrac{d}{dT} \ln T$

$$= NkT = nRT$$

$$u = \frac{U}{n} = RT$$

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v = R$$

E & M – Solution.

a) From Gauss's law $\vec{D} = \hat{e}_r\, Q/r^2$ where Q is the total charge on the inner electrode and $a < r < b$. Then $\vec{E} = \vec{D}/\varepsilon = \hat{e}_r\, Q/\varepsilon r^2$, and $\vec{J} = \sigma \vec{E} = \hat{e}_r\, K Q/\varepsilon r^3$. Conservation of current yields

$$\nabla \cdot J = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 J_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{QK}{\varepsilon r}\right) = 0$$

so $\varepsilon r = \bar{\varepsilon}$ (a const.) and $\varepsilon(r) = \bar{\varepsilon}/r$.

b)
$$\vec{E} = \frac{\hat{e}_r Q}{\bar{\varepsilon} r} = -\vec{\nabla}\phi = -\frac{d\phi}{dr} \quad \text{so}$$

$$\phi(r) = -\frac{Q}{\bar{\varepsilon}}\int_a^r \frac{dr'}{r'} = -\frac{Q}{\bar{\varepsilon}} \ln\frac{r}{a} + C$$

$$\phi(b) = -\frac{Q}{\bar{\varepsilon}} \ln(b/a) + C = \phi_0$$

$$\phi(r) = \phi_0 + \frac{Q}{\bar{\varepsilon}} \ln(b/r)$$

$$\Delta\phi = \phi(b) - \phi(a) = \frac{Q}{\bar{\varepsilon}} \ln(b/a)$$

The total resistance R between the two spheres:

$$R = \int \frac{dr}{4\pi r^2 \sigma} = \int_a^b \frac{dr}{4\pi K r} = \frac{1}{4\pi K} \ln\frac{b}{a}$$

$$I = \Delta\phi/R = \frac{Q}{\bar{\varepsilon}} \cdot 4\pi K \quad \text{or} \quad \frac{Q}{\bar{\varepsilon}} = \frac{I}{4\pi K}$$

$$\phi(r) = \phi_0 + \frac{I}{4\pi K} \ln(b/r)$$