PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #35

January 5, 1980

General Instructions

This Comprehensive Examination for Winter 1980 (#35) consists of six problems of equal weight (20 points each). Please check that you have all of them.

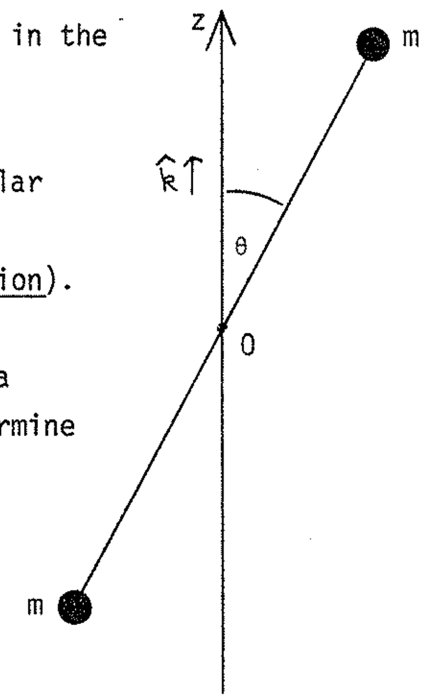
Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you *may* find useful. Please return it at the end of the exam.

$$\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}du = (n-1)\Gamma(n-1) \equiv (n-1)!$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

1. Consider the two equal masses (mass points) shown in the figure to be connected by a rigid massless rod.



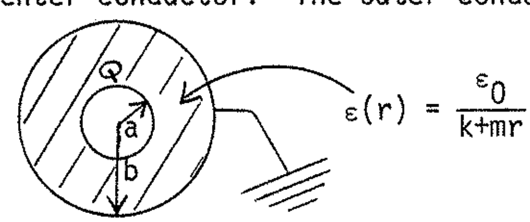
(10) (a) If the system is rotating with constant angular velocity $\vec{\omega} = \omega\hat{k}$, determine the total angular momentum of the system (magnitude and direction).

(10) (b) Construct a solid sphere of radius R out of a collection of such mass points and thus determine the angular momentum of this sphere rotating with angular velocity $\vec{\omega}$. (Calculate the magnitude of $\vec{L}$ and indicate why its direction is parallel to $\vec{\omega}$.)

2. The region between two concentric, conducting spheres of radii a and b is filled by an inhomogeneous dielectric with

$$\varepsilon = \varepsilon_0/(k+mr).$$

A charge Q is placed on the center conductor. The outer conductor is connected to ground.



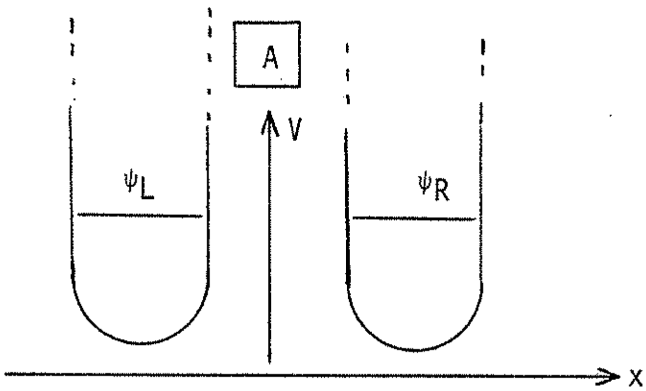
Derive expressions for:

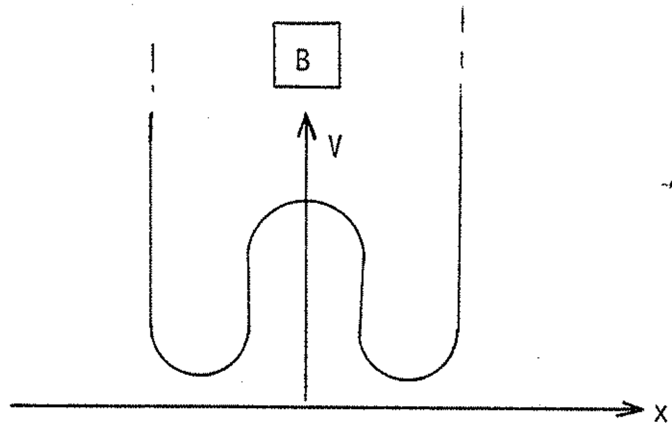
(4) (a) The displacement $\vec{D}$ at any point between the spheres;

(8) (b) The capacitance between the two electrodes;

(8) (c) The volume density of polarization charge at any radius r between the spheres.

3. There are two, normalized stationary states ψ_L and ψ_R which are, respectively, the solutions for two, identical, unconnected potential wells placed symmetrically about the origin (Figure A).





The Hamiltonian in either well is H_0:

$$H_0\psi_L = E_0\psi_L \quad , \qquad H_0\psi_R = E_0\psi_R$$

$$<\psi_L|H_0|\psi_R> = <\psi_R|H_0|\psi_L> = 0.$$

For the rest of this problem, assume these are the only 2 levels present and that a symmetric interaction, V, is now introduced which weakly connected the two wells as in B.

(2) (a) Why must the new Hamiltonian, $H = H_0 + V$, be invariant under the parity (reflection) operator P?

(2) (b) What is the effect of P on ψ_L and ψ_R?

(4) (c) Find two, normalized states, which are eigenstates of P. Indicate their eigenvalues (parity) and verify their normalization.

(2) (d) Why can there be no transitions between the states of part (c)?

(8) (e) Find the two energies of this system. Express your answers in terms of $<\psi_L|H_0|\psi_L>$, $<\psi_L|V|\psi_L>$ and $<\psi_L|V|\psi_R>$.

(2) (f) If the particle is in the left well at time t = 0, what is probability to be in the right well at time t?

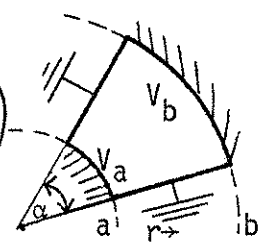
4. Consider a cubical box of edge L to contain an ideal quantum gas of N indistinguishable particles at absolute temperature T. Conditions are such that the gas can be characterized by Boltzmann statistics.

(15) (a) Derive an expression for the partition function of this gas in terms of volume (V) and temperature (T).

(5) (b) Derive an expression for the chemical potential for this system.

5. A two-dimensional cavity is made up of concentric circles and radii as shown below. The angle α is general and no dependence on a third dimension is to be assumed for any functions. The straight sides of the cavity are grounded and the potential on the curved sides is given by $V_a(\theta)$ and $V_b(\theta)$.

(for the potential in the cavity)

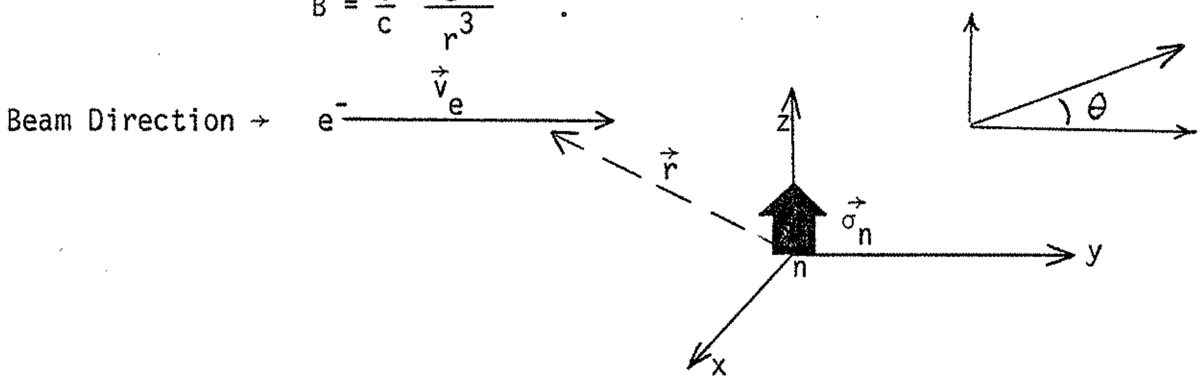


(10) (a) Give an expansion, in terms of separate functions of r and θ, which satisfies explicitly the boundary condition on the straight sides.

(10) (b) Express the expansion coefficients in terms of definite integrals involving the prescribed surface potentials.

6. Consider the scattering of non-relativistic electrons from a polarized, static neutron. Since the neutron has spin ½ it has a magnetic moment $\vec{\mu} = \beta\vec{\sigma}$, and the scattering is caused by the interactions of this moment with the magnetic field of the passing electron,

$$\vec{B} = \frac{e}{c}\,\frac{\vec{v}_e \times \vec{r}}{r^3} \quad .$$



(6) (a) What is the quantum mechanical interaction "potential" (or Hamiltonian) <u>operator</u> for the electron-neutron interaction?

( ) (b) An electron of energy $\hbar^2 k_0^2/2m$, incident in +y direction, is scattered by an angle θ in the xy plane. What are the energy and momentum of the electron in the final state?

(7) (c) Evaluate, in lowest order perturbation theory, the matrix element of the interaction potential between initial and final states, $\langle\psi_f|V|\psi_i\rangle$.

(4) (d) If, in the Born Approximation, the differential cross section is

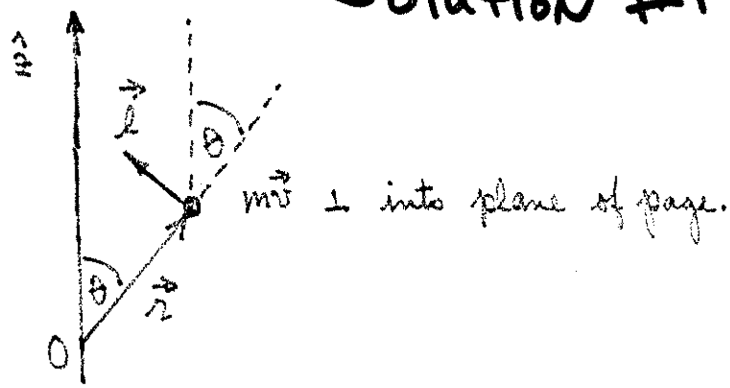
$$d\sigma/d\Omega = \alpha|\langle\psi_f|V|\psi_i\rangle|^2 \quad ,$$

what is the cross section for electron-neutron scattering?

#1.

Solution #1

(a)



Consider the upper of the two masses
$\vec{l} = \vec{r} \times m\vec{v}$ where $\vec{v} = \vec{\omega} \times \vec{r}$

$\therefore \quad \vec{l} = \vec{r} \times (m\,\vec{\omega} \times \vec{r})$

Magnitude of $\vec{l}$ is $\quad l = m r^2 \omega \sin\theta$

One gets an identical contribution from the second mass, and so

$$l_{Total} = 2 m r^2 \omega \sin\theta$$

The orientation of $\vec{l}$ is, as indicated above, given by the cross product rule. $\vec{l}$ makes an angle θ above the x-y plane and precesses about the z-axis at angular speed ω.

#1 (Contd)

(a) ALTERNATE METHOD.

If you want to use I, the moment of inertia, you must realize it is a tensor (dyadic).

$$L_i = I_{ij}\,\omega_j\ , \quad \text{where } \omega_j = (0, 0, \omega) \text{ in our case.}$$

$$\therefore \quad L_x = I_{xz}\,\omega$$
$$L_y = I_{yz}\,\omega$$
$$L_z = I_{zz}\,\omega$$

where $I_{xz} = -\sum_i m_i x_i z_i = -2mxz$

$$I_{yz} = -\sum_i m_i y_i z_i = -2myz$$

and $I_{zz} = \sum_i m_i (x_i^2 + y_i^2) = 2m(x^2+y^2)$

where x, y and z are the coordinates of one of our two particles.

$$\therefore \quad L_x = -2mr^2\omega \cos\theta \sin\theta \cos\phi$$
$$L_y = -2mr^2\omega \cos\theta \sin\theta \sin\phi$$
$$L_z = 2mr^2\omega \sin^2\theta.$$

where $x = r\sin\theta\cos\phi$

$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta \ .$$

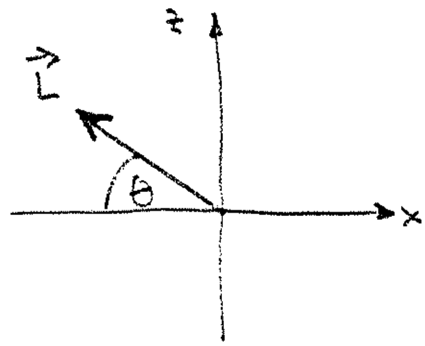
Here $\phi = \phi_0 + \omega t$, ϕ_0 being determined by initial conditions.

#1 (Cont)

Magnitude of L : $L = \sqrt{L_x^2 + L_y^2 + L_z^2}$

Doing the straightforward algebra and utilizing the fact that $\cos^2\alpha + \sin^2\alpha = 1$, yields

$$L = 2mr^2\omega\sin\theta .$$



At $t=0$, say $\phi = 0$ $(\phi_0 = 0)$

$$L_x(0) = -2mr^2\omega\cos\theta\sin\theta$$
$$L_y(0) = 0$$
$$L_z(0) = 2mr^2(\sin^2\theta)\omega$$

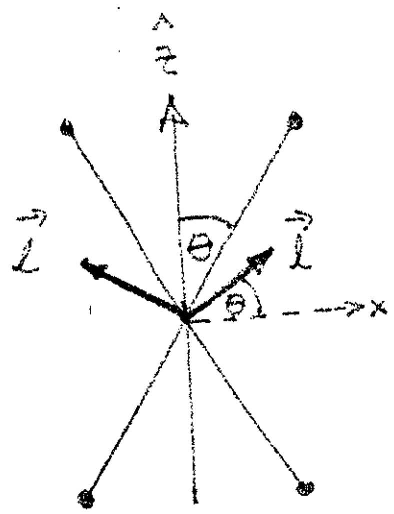
Angle $\vec{L}$ makes with x-axis is, say β

$$\tan\beta = \frac{L_z(0)}{L_x(0)} = -\tan\theta$$

so $\vec{L}$ precesses about z at fixed angle θ with respect to x-y plane as indicated in simpler approach.

#1

(b)



By symmetry we see that appropriate pairs of dumbells produce a cancelation of all but the z-component of the angular momentum.

i.e.

$$L_{sphere} = \iiint dl_z \text{ (of dumbell)}$$

From part (a) we have $l = 2mr^2\omega \sin\theta$

$$\therefore \quad dl_z = 2r^2\omega \sin^2\theta \, dm$$

where $dm = \rho r^2 \sin\theta \, dr \, d\theta \, d\varphi$

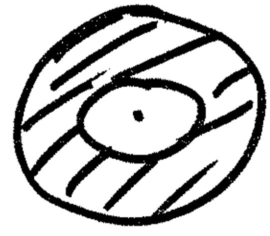
$$\therefore \quad L_{sphere} = 2\omega\rho \int_0^R r^4 dr \int_0^{\pi/2} \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= 2\omega\rho \cdot \frac{R^5}{5} \cdot \frac{2}{3} \cdot 2\pi$$

$$L_{sphere} = \frac{2}{5} MR^2\omega \quad \text{and} \quad \vec{L}_{sphere} \parallel \hat{z}.$$

EM/ 6-U6

Solution #2



(a) D

$$\bar{\nabla}\cdot\bar{D} = 4\pi\rho$$

$$\Rightarrow \int \nabla\cdot D \, d^3x = 4\pi \int \rho \, d^3x$$

$$\int D\cdot dA = 4\pi Q$$

$$4\pi r^2 D = 4\pi Q$$

$$\boxed{D = Q/r^2}$$

(b) C

$$C = Q/V \quad , \quad Q \text{ on the outside}$$

$$V = -\int E\cdot dr$$

$$D = \epsilon E \quad , \quad E = D/\epsilon$$

$$E = \frac{1}{\epsilon} D = \frac{Q(k+mr)}{\varepsilon_0 r^2}$$

$$\int_a^b E\cdot dr = \frac{Q}{\varepsilon_0}\int_a^b \frac{(k+mr)}{r^2}\,dr = \frac{Q}{\varepsilon_0}\left[k\int_a^b \frac{dr}{r^2} + m\int_a^b \frac{dr}{r}\right]$$

$$= \frac{Q}{\varepsilon_0}\left[\frac{k}{a} - \frac{k}{b} + m\ln\frac{b}{a}\right]$$

$$C = Q/V = \frac{\varepsilon_0}{\frac{k}{a} - \frac{k}{b} + m\ln\frac{b}{a}} = \frac{\varepsilon_0}{\frac{k(b-a)}{ab} + m\ln b/a}$$

$$\boxed{C = \frac{\varepsilon_0 ab}{k(b-a) + m\,ab\ln b/a}}$$

2 (cont)

(c) Volume Polarization

$\underset{\sim}{D}$ accounts ~~for polarization~~ via

$$D = \varepsilon E = E + 4\pi \underset{\sim}{P}$$

$$\nabla\cdot\underset{\sim}{E} = 4\pi\rho_{TOTAL}$$
$$\nabla\cdot D = 4\pi\rho_{free}$$

$$\nabla\cdot\underset{\sim}{E} - \nabla\cdot\underset{\sim}{D} = 4\pi(\rho_{TOT} - \rho_{free}) = 4\pi\rho_{polar}$$

$$\frac{\nabla\cdot(E-D)}{4\pi} = \rho_{pol} = \nabla\cdot(\underset{\sim}{E}(1-\varepsilon))$$

$$\bar{E} = \bar{D}/\varepsilon = \frac{Q(k+mr)}{\varepsilon_0 r^2}$$

$$1-\varepsilon = 1 - \frac{\varepsilon_0}{k+mr} = \frac{k+mr-\varepsilon_0}{k+mr}$$

$$\nabla\cdot E = \frac{1}{\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}(Q(k+mr)) = \frac{Qm}{\varepsilon_0 r^2}$$

$$\nabla(1-\varepsilon) = -\nabla\varepsilon = -\hat{\varepsilon}_r\frac{\varepsilon M}{(k+mr)^2}$$

$$\rho_p = \frac{1}{4\pi}(\nabla\cdot\underset{\sim}{E} - \nabla\cdot D) = \frac{1}{4\pi}\nabla\cdot(\underset{\sim}{E}(1-\varepsilon)) = \frac{(1-\varepsilon)\nabla\cdot\underset{\sim}{E} + E\cdot\nabla(1-\varepsilon)}{4\pi}$$

$$= \frac{1}{4\pi}\left\{\frac{k+mr-\varepsilon_0}{k+mr}\frac{QM}{\varepsilon_0 r^2} + \frac{\varepsilon_0 M}{(k+mr)^2}\frac{Q(k+mr)}{\varepsilon_0 r^2}\right\}$$

$$\rho_p = \frac{1}{4\pi}\frac{QM}{\varepsilon_0 r^2}\left\{\frac{k+mr-\varepsilon_0 - \varepsilon}{k+mr}\right\} = \frac{1}{4\pi}\frac{QM}{\varepsilon_0 r^2}$$

there must be an easier way

Solution #3

(2) ⓐ $H = H_0 + V$ Since the wells are symmetrical about the origin, a reflection through the origin leads to a physically equivalent problem, i.e. H must be invariant under reflection.

(2) ⓑ $P\psi_L = \psi_R$
$P\psi_R = \psi_L$ } i.e. switch

(4) ⓒ $\psi_A = \frac{1}{\sqrt{2}}(\psi_L - \psi_R)$ $\quad\quad$ $\psi_S = \frac{1}{\sqrt{2}}(\psi_L + \psi_R)$

$$\underbrace{P\psi_A = \tfrac{1}{\sqrt{2}}(\psi_R - \psi_L) = -\psi_A}_{\text{Parity } -}, \qquad \underbrace{P\psi_S = \tfrac{1}{\sqrt{2}}(\psi_R + \psi_L) = \psi_S}_{\text{Parity } +}$$

$$\langle\psi_A|\psi_A\rangle = \tfrac{1}{2}\{\langle\psi_L|\psi_L\rangle + \langle\psi_R|\psi_R\rangle - 2\langle\psi_L|\psi_R\rangle\}$$
$$= \tfrac{1}{2}\{1 + 1 - 0\} = 1 \checkmark$$

ⓔ $H = H_0 + V$

$$E_A = \langle\psi_A|H_0+V|\psi_A\rangle = \tfrac{1}{2}\{\langle\psi_L|H_0+V|\psi_L\rangle - \langle\psi_L|H_0+V|\psi_R\rangle + \langle\psi_R|H_0+V|\psi_R\rangle - \langle\psi_R|H_0+V|\psi_L\rangle\}$$

let $E_0 = \langle\psi_L|H_0|\psi_L\rangle = \langle\psi_R|H_0|\psi_R\rangle$ (the same)
$E' = \langle\psi_L|V|\psi_R\rangle = \langle\psi_R|V|\psi_L\rangle$ (by symmetry)
$\langle\psi_L|H_0|\psi_R\rangle = \langle\psi_R|H_0|\psi_L\rangle = 0$: no connect
$\bar{V} = \langle\psi_L|V|\psi_L\rangle = \langle\psi_R|V|\psi_R\rangle$

~~$E_A = \frac{1}{2}\{E_0$~~

$$E_A = \tfrac{1}{2}\{E_0 + \bar{V} - E' + E_0 + \bar{V} - E'\}$$

$$\longrightarrow E_A = E_0 + \bar{V} - E'$$

$$E_S = \langle\psi_S|H_0+V|\psi_S\rangle = \tfrac{1}{2}\{E_0 + \bar{V} + E' + E_0 + \bar{V} + E'\}$$

$$\longrightarrow E_S = E_0 + \bar{V} + E'$$

3 (Contd)

(2) d Since the states have different parity and V is symmetric, $\langle \psi_S | V | \psi_A \rangle = \langle \psi_A | V | \psi_S \rangle = 0$.

(2) F $t=0$, $\psi = \psi(0)$. ψ_A & ψ_S are eigenstates

get $\quad \psi_L = \frac{1}{\sqrt{2}}(\psi_A + \psi_S)$

so $\psi_L(t) = \frac{1}{\sqrt{2}}\left(\psi_A(0)\, e^{-iE_A t/\hbar} + \psi_S(0)\, e^{-iE_S t/\hbar} \right)$

get $\psi_A = \frac{1}{\sqrt{2}}(\psi_L - \psi_R)$, $\psi_S = \frac{1}{\sqrt{2}}(\psi_L + \psi_R)$

$$\psi_L(t) = \frac{1}{2}\left[\psi_L\left(e^{-iE_A t/\hbar} + e^{-iE_S t/\hbar}\right) + \psi_R\left(e^{-iE_S t/\hbar} - e^{-iE_A t/\hbar}\right)\right]$$

$$\text{Prob for } \psi_L = \left| \frac{1}{2}\left(e^{-iE_A t} + e^{-iE_S t}\right)\right|^2$$

$$= 1/4\left(1 + e^{i(E_A - E_S)t} + e^{i(E_S - E_A)t} + 1\right)$$

$$\text{Prob}(\psi_L) = \frac{1}{2}\left(1 + \cos[(E_A - E_S)t/\hbar]\right)$$

$$\text{Prob}(\psi_R) = \frac{1}{4}\left(2 - 2\cos(E_S - E_A)t/\hbar\right)$$

$$= \cancel{\tfrac{4}{2}\sin(E_S - E_A)t/\hbar}$$

$$\frac{1}{2}\left(1 - \cos(E_S - E_A)t/\hbar\right)$$

Solution #4

#4

$$Z_{MB} = \sum_i e^{-\beta \varepsilon_i} \quad \text{for each particle.}$$

Since the N particles are independent the partition function for all N is given by

$$Z_{MB} = Z_1 \cdot Z_2 \cdot \cdots \cdot Z_N = \left(\sum_i e^{-\beta \varepsilon_i} \right)^N$$

[since also the same set of single particle energy levels are available to each particle.]

The particles are also indistinguishable and to take this into account we need to divide the above expression by N!

$$\therefore \quad Z = \frac{Z_{MB}}{N!} = \frac{\xi^N}{N!}$$

Here $$\xi = \sum_{k_x, k_y, k_z} e^{-\beta \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)}$$

$$\xi = \left(\sum_{k_x} e^{-\beta \frac{\hbar^2}{2m} k_x^2} \right) \left(\sum_{k_y} e^{-\beta \frac{\hbar^2}{2m} k_y^2} \right) \left(\sum_{k_z} e^{-\beta \frac{\hbar^2}{2m} k_z^2} \right).$$

Boundary conditions yield

$$k_i = \frac{2\pi}{L} n_i \qquad (i \equiv x, y, z).$$

#4

Going over from $\Sigma \to \int$

$$\xi = \left(\int_{-\infty}^{\infty} e^{-\frac{\beta \hbar^2}{2m} k_x^2} \left(\frac{L_x}{2\pi}\right) dk_x \right) \left(\cdots \right) \left(\cdots \right)$$

$\uparrow$ density of states

$$\xi = \frac{L}{\pi} \left(\frac{m}{2\beta}\right)^{1/2} \frac{1}{\hbar} \underbrace{\int_0^{\infty} e^{-u} u^{-1/2} du}_{\Gamma(\frac{1}{2}) = \sqrt{\pi}} \times \left(\quad \right) \times \left(\quad \right)$$

$$\therefore \quad \xi = \frac{L^3}{\hbar^3} \left(\frac{mkT}{2\pi}\right)^{3/2} = \frac{V}{h^3} \left(2\pi mkT\right)^{3/2}$$

$$\therefore \quad Z = \frac{V^N (2\pi mkT)^{3N/2}}{h^{3N} N!} .$$

$$F = -kT \ln Z \qquad \text{(Free Energy).}$$

and $$\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T \text{ const.}} \qquad \text{(The Chem. Potential).}$$

note that $$\frac{\mu}{-kT} = \frac{\partial}{\partial N} \left\{ N \ln \xi - (N \ln N - N) \right\}$$

making use of Stirling's approximation for $\ln N!$

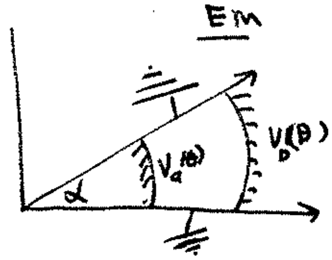
#4

$$\therefore \quad \mu = -kT \left\{ \ln \Xi - \left[N/N + \ln N - 1 \right] \right\}$$

$$= -kT \log\left(\frac{\Xi}{N}\right)$$

$$\mu = -kT \log \left\{ \frac{V\,(2\pi m kT)^{3/2}}{N\,h^3} \right\}.$$

EM



Solution #5

Cylindrical Coordinates (no z yet)

let: $\Phi(r\theta) = R(r)\,Q(\theta)$

(1) θ Dependence

$$\nabla^2 \Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)Q + R\frac{1}{r^2}\frac{\partial^2 Q}{\partial \theta^2}$$

$$d^2Q/d\theta^2 + \nu^2 Q = 0 \qquad \frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}{R} + \frac{\frac{1}{r^2}\frac{d^2Q}{d\theta^2}}{Q} = 0$$

$$Q(\theta) = e^{\pm i\nu\theta}$$

B.C. : $Q(0) = Q(\alpha) = 0$

$\therefore$ take linear combinations,

$$* \quad \boxed{Q(\theta) = \sin\frac{\pi n}{\alpha}\theta} = \sin\nu\theta$$

$$(\nu = \pi n/\alpha)$$

(2) r-dependence

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{\nu^2}{r^2}R = 0$$

try $R = r^m$

$$m(m-1)\,r^{m-2} + m\,r^{m-2} - \nu^2 r^{m-2} = 0$$

$$m^2 - m + m - \nu^2 = 0$$

$$m^2 = \nu^2$$

$$m = \pm\nu = \pm\pi n/\alpha$$

$$\Phi = \sin\nu\theta\left[a r^{\nu} + b r^{-\nu}\right] \quad \text{Ans (1)} \quad \boxed{(\nu = \pi n/\alpha)}$$

Most General Solution: $\boxed{\Phi = \sum_{n=1}^{\infty} \sin\left(\frac{\pi n\theta}{\alpha}\right)\left[A_n r^{\nu} + B_n r^{-\nu}\right]}$

(2)

$$\Phi(r=a) = \sum_n \sin\frac{\pi n\theta}{\alpha}\left[A_n a^{\nu} + B_n a^{-\nu}\right] = V_a(\theta)$$

mult. by $\sin\frac{\pi n\theta}{\alpha}$ and integrate over $\theta\Big|_0^{\alpha}$; and likewise for $V_b(\theta)$

$$I = \int_0^{\alpha} V_a(\theta)\sin\frac{m\pi\theta}{\alpha}\,d\theta = \frac{\alpha}{2}\left[A_n a^{\nu} + B_n a^{-\nu}\right]$$

$$J = \int_0^{\alpha} V_b(\theta)\sin\frac{m\pi\theta}{\alpha}\,d\theta = \frac{\alpha}{2}\left[A_n b^{\nu} + B_n b^{-\nu}\right]$$

now solve

$$\boxed{A_n = \frac{a^{\nu}\frac{2}{\alpha}I - b^{\nu}\frac{2}{\alpha}J}{a^{2\nu} - b^{2\nu}}, \quad B_n = \frac{a^{-\nu}\frac{2}{\alpha}I - b^{-\nu}\frac{2}{\alpha}J}{a^{-2\nu} - b^{-2\nu}}}$$

#6 Solution

The dipole interaction $V = -\vec{\mu}\cdot\vec{B}$

a) $\vec{\mu} = \beta\vec{\sigma}$

$$B = \frac{e}{c}\,\frac{\vec{v}\times\vec{r}}{r^3}$$

and $\vec{v} \Rightarrow \frac{1}{m}\vec{p} \Rightarrow -\frac{i}{m}\hbar\vec{\nabla}$

$$V = \frac{e\hbar\beta}{mcr^3}\;\vec{\sigma}\cdot\vec{r}\times\frac{1}{i}\vec{\nabla}$$

b) Since the neutron is static it carries no kinetic energy and

initial electron K.E. = final K.E. $= \hbar^2 k_0^2/2m$

$$|\vec{k_0}| = |\vec{k_f}|$$

c) For wave functions take

$$\psi_I = \frac{1}{\sqrt{L^3}}\chi_I e^{i\vec{k_0}\cdot\vec{r}} \qquad \psi_F = \frac{1}{\sqrt{L^3}}\chi_F e^{i\vec{k_f}\cdot\vec{r}}$$

$$\langle\psi_F|V|\psi_I\rangle = \frac{e\hbar\beta}{mcL^3}\langle\chi_F|\vec{\sigma}|\chi_I\rangle\cdot$$

$$\int e^{-i\vec{k_f}\cdot\vec{r}}\left(\vec{r}\times\frac{1}{i}\vec{\nabla}\right)e^{i\vec{k_0}\cdot\vec{r}}\,\frac{d^3r}{r^3}$$

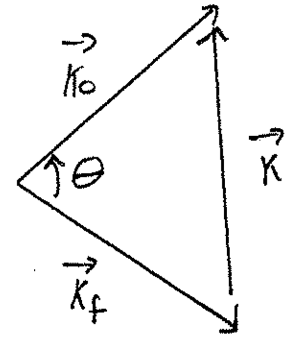
$$= \frac{e\hbar\beta}{mcL^3}\langle\chi_F|\vec{\sigma}|\chi_I\rangle\cdot\int e^{i\vec{K}\cdot\vec{r}}\,(\hat{e}_r\times\vec{k_0})\,dr\,d(\cos\theta_r)\,d\phi_r$$

$\chi_{I,F}$ are the spin wave functions

$$\vec{K} \equiv \vec{k_0} - \vec{k_f} = 2|\vec{k_0}|\sin\theta/2$$

#6 (Contd)

and θ is ∡ between $\vec{K}_o$ and $\vec{K}_f$



To do the integral choose the polar axis in the direction of $\vec{K}$ so that $e^{i\vec{K}\cdot\vec{r}}$ is independent of ϕ.

In doing the ϕ integration all components of $\vec{e}_r \times \vec{K}_o$ average to zero except the comp. which is $\perp$ to $\vec{K}_o$, $\vec{K}_f$, and $\vec{K}$. Call this direction

$$\vec{n} = \frac{\vec{K}_o \times \vec{K}_f}{|\vec{K}_o \times \vec{K}_f|}$$

$$\int d\phi \, (\hat{e}_r \times \vec{K}_o) = 2\pi K_o \cos\theta_{Kr} \cos\theta/2 \; \hat{n}$$

so
$$\langle \psi_F | v | \psi_I \rangle = \frac{e\hbar\beta}{mcL^3} (2\pi) K_o \int \cos\theta_{Kr} \, e^{i\vec{K}\cdot\vec{r}} \, dr \, d(\cos\theta_{Kr})$$
$$\times \cos\theta/2 \; \langle \chi_F | \vec{\sigma}_z | \chi_I \rangle \cdot \hat{n}$$

$$= \frac{i(2\pi)e\hbar\beta}{mcL^3} \cot\theta/2 \; \langle \chi_F | \vec{\sigma} | \chi_I \rangle \cdot \hat{n}$$

d) To calculate a cross section we must average over initial spins & sum over final electron spins. Thus (dropping constants)

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{2} \sum_{m_i, m_f} |\langle \psi_F | V | \psi_I \rangle|$$

#6 Contd

or $$\frac{d\sigma}{d\Omega} \propto \mathrm{Tr}\left[\vec{\sigma}\cdot\hat{n}\ \vec{\sigma}\cdot\hat{n}\right] \cot^2\theta/2$$

$$\propto \cos^2\theta_n \cot^2\theta/2$$

where θ_n is $\measuredangle$ between $\hat{n}$ and z axis.

#

d) c The polarization charge density is found from

$$D = \varepsilon E = E + 4\pi P$$

$$\nabla \cdot E = 4\pi \rho_{Total}$$

$$\nabla \cdot D = 4\pi \rho_{free}$$

$$4\pi \rho_{pol.} = 4\pi (\rho_{total} - \rho_{free}) = \nabla \cdot E - \nabla \cdot D$$

$$\nabla \cdot D = \nabla \cdot (q/r^2) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \, q/r^2) = C = 4\pi \rho_{free}$$

$$\nabla \cdot E = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{q}{\varepsilon r} \right) = \frac{q}{\varepsilon r^2} = 4\pi \rho_{Total}$$

So

$$\rho_{pol} = \frac{q}{4\pi \varepsilon r^2} = \frac{I}{4\pi \cancel{K}} \times \frac{1}{4\pi r^2}$$