PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #33

March 31, 1979

General Instructions

This Comprehensive Examination for Spring 1979 (#33) consists of six problems of equal weight (20 points each). Half of the material is judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show as clearly as possible all your steps so that partial credit can be given liberally in case you do not complete a problem or make errors. Use no scratch paper; do all work in the bluebook, using one bluebook per problem. Use the exam paper for question #3, and make sure before you turn it in that it has your code number.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

For a harmonic oscillator:

$$\begin{cases} \langle n|x|n'\rangle = \sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{n+1}{2}}\,\delta_{n+1,n'} + \sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{n}{2}}\,\delta_{n-1,n'} \\ \langle n|p|n'\rangle = -i\sqrt{m\hbar\omega}\sqrt{\frac{n+1}{2}}\,\delta_{n+1,n'} + i\sqrt{m\hbar\omega}\sqrt{\frac{n}{2}}\,\delta_{n-1,n'} \end{cases}$$

$$E_n' = E_n + \langle n|V|n\rangle + \sum_{m\neq n}\frac{|\langle n|V|m\rangle|^2}{E_n-E_m}$$

$$J^{\pm}|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\;|m\pm 1\rangle$$

$$[J_x,J_y] = i\hbar J_z$$

$$U = \sum_{n=1} A_n r^{-n} e^{in\theta} + \sum_{n=0} B_n r^n e^{in\theta} + A_0 \ln r$$

$$U = \sum_{n=0} A_n r^n P_n(\cos\theta) + \sum_{n=0} B_n r^{-(n+1)} P_n(\cos\theta)$$

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{\ell=0} P_\ell(z)t^\ell \qquad P_\ell^m(z) = (z^2-1)^{\frac{1}{2}m}\,\frac{d^m P_\ell(z)}{dz^m}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{-\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right] \qquad \vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left[\frac{\vec{M}\times(\vec{r}-\vec{r}')}{|r-r'|^3}\right]dv'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\times(\vec{r}'-\vec{r})}{|r'-r|^3}\,dv' \qquad \vec{B} = \frac{\mu_0 I}{2\pi a}\,\vec{k}$$

$$\sum_{m_1, m_2} \langle m_1 m_2 | jm \rangle \langle m_1 m_2 | j'm' \rangle = \delta_{jj'} \delta_{mm'}$$

$$|jm\rangle = \sum_{m=m_1+m_2} |m_1 m_2\rangle \langle m_1 m_2 | jm \rangle$$

$$\psi_{n\ell m} = R_{n\ell}(r)\, \Theta_{\ell,m}(\theta)\, \Phi_m(\phi) \qquad \langle R_{n\ell} | R_{n'\ell'} \rangle = \delta_{nn'}\, \delta_{\ell\ell'}$$

$$\langle \Theta_{\ell' m'} | \Theta_{\ell m} \rangle = \delta_{\ell\ell'}\, \delta_{mm'} \qquad \langle \Phi_m | \Phi_{m'} \rangle = \delta_{mm'}$$

$$\langle \Theta_{\ell,m} | \cos\theta | \Theta_{\ell',m'} \rangle = \delta_{\ell,\ell'\pm 1}\, \delta_{|m|,|m'|}\, f(\ell,|m|)$$

$$h = 6.6 \cdot 10^{-34} \text{ joule-sec} = 4.14 \cdot 10^{-15} \text{ eV-sec}$$

$$a_0 = h^2/m_e e^2 = 0.529 \text{ Å}$$

$$c = 3 \times 10^8 \text{ m/sec}$$

$$k = 1.38 \times 10^{-23} \text{ joule/°K}$$

$$E_0 = m_e e^4/2h^2 = 13.6 \text{ eV}$$

$$\psi_{100} = \pi^{-1/2}(a_0)^{-3/2} \exp(-r/a_0)$$

$$\nabla \times \phi\vec{A} = \phi \nabla \times \vec{A} + (\nabla\phi) \times \vec{A}$$

$$\nabla \cdot \phi\vec{A} = \phi \nabla \cdot \vec{A} + \nabla\phi \cdot \vec{A}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\frac{d\vec{A}}{ds} = \left(\frac{d\vec{r}}{ds} \cdot \nabla \right) \vec{A}$$

$$\nabla = \frac{\vec{r}}{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \theta}$$

$$\ln n! \approx \frac{1}{2} \ln 2\pi n + n \ln n - n$$

$$(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} x^k$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

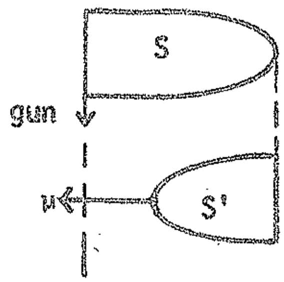
$$2\pi\delta(q-q') = \int_{-\infty}^{\infty} e^{-i(q-q')s} ds$$

1. A single-electron atom is in an orbital p state. Its total angular momentum quantum number j is 1/2.

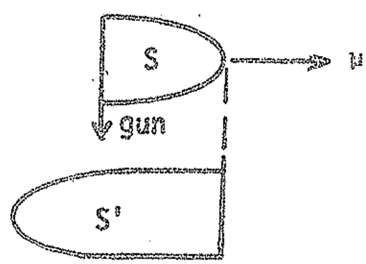
(15) (a) Derive the relations connecting the product wave functions of the spin and orbital components of this system with the wave functions of the total angular momentum.

(5) (b) Prove that $|\psi|^2$ is spherically symmetric.

2. Two hostile spaceships, S and S', approach each other on a parallel near-collision course at relativistic velocities. S and S' have the same proper length L and their relative velocity is μ. A laser gun protrudes from the tail of S in a direction perpendicular to the direction of the relative motion. When the pilot in the front tip of S is even with the tail of S', he fires his gun. However from his point of view, the length of S' is contracted and the laser beam passes harmlessly over the bow of the enemy ship as shown in the sketch below.



But from the point of view of S', it is S whose length is contracted, and the laser beam hits it broadside as shown below.



(a) Has S' been hit or not? If it has been hit, find the point at which it has been hit. If it has not been hit, calculate the distance by which it has been missed. You should assume that the two ships pass so closely that the propagation time of the laser beam can be neglected.

(b) Explain carefully the origin of the apparent paradox. Sketch the relative positions of the two ships as seen by S' at the moment of firing. Give the coordinates of the fore and aft of each ship in terms of some convenient set of axes.

3. Complete the following sentences in the space provided:

(a) By scattering electrons from a crystal, Davisson and Germer provided evidence for ______________________________

______________________________.

(b) The energy of an electron knocked out of a metal by a beam of light depends on

(i) ______________________________,

(ii) ______________________________,

(iii) ______________________________,

but not on

(iv) ______________________________.

This is evidence for the quantization of ______________________________

____

(c) Einstein and Debye applied Planck's hypothesis to the specific heat of solids and were able to explain

____

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(continued on next page)

3. (continued)

(d) Give an approximate value in cm for the size of

(i) A hydrogen atom.

(ii) A lead atom.

(iii) Hydrogen nucleus (proton).

(iv) Lead nucleus.

(v) The universe.

4. A rigid rotator with the moment of inertia I_0 has a magnetic moment $\vec{\mu} = \mu_0 \vec{L}$ which interacts with a magnetic field $\vec{B}$ contributing an extra term $H_m = \mu_0 \vec{L} \cdot \vec{B}$ to the Hamiltonian.

(10) (a) Suppose $\vec{B}$ is uniform in the x direction

(i) Which of the following quantities are constants of the motion: L_x, L_y, L_z, L^2, parity. (Parity sends $\vec{B}$ into $\vec{B}$.)

(ii) What are the energy levels of this rotator and what is the degeneracy of each level for the two cases: $\vec{B} = 0$ and $\vec{B} \neq 0$.

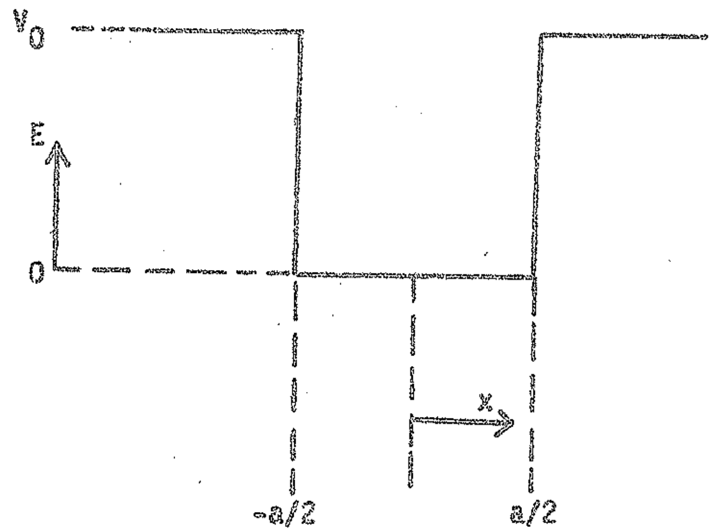
(b) Suppose $\vec{B} = 0$ and the wave function of the rotator at time $t = 0$ is

$$\psi(\theta, \phi, t=0) = \frac{1}{4}\sqrt{\frac{3}{\pi}}\,(1 + \cos\theta)$$

(i) For all possible values of the quantum numbers (ℓ, m) give the probability that the rotator has an angular momentum $\ell\hbar$ with a z component $m\hbar$.

(ii) What is the wave function at a later time $t > 0$?

5. A particle of mass m is in a one-dimensional square well with dimensions shown below. It is in an eigenstate at an energy E_1 whose wave function has two nodes.



(a) How many eigenstates are there with energies less than E_1?

(b) Estimate how many eigenstates there are with energies E such that $E_1 < E < V_0$.

(c) Evaluate the ratio of probability densities $\rho(\frac{a}{2})/\rho(0)$ for the eigenstate in terms of m, a, V_0, and E_1.

(d) Evaluate the ratio of probability densities $\rho(a)/\rho(0)$ for the eigenstate in terms of m, a, V_0, and E_1.

6. An isolated two-level system with a population inversion is commonly described as being at a negative temperature.

(5) (a) Using the first and second laws of thermodynamics, show that heat will always flow spontaneously from a system at a negative temperature into a system at a positive temperature if they are placed in thermal contact.

(10) (b) Consider an isolated system of N particles each of which has one of two possible states. N_1 of them are in a state at energy $E_1 = -\varepsilon$ and N_2 are in a state at an energy $E_2 = +\varepsilon$. The particles are at thermal equilibrium with each other, so they have a temperature T which may be negative. Treating the system as a microcanonical ensemble with an internal energy $U = N_1E_1 + N_2E_2$ and an entropy

$$S = k \ln \frac{N!}{N_1! \ N_2!}$$

(i) Derive the temperature T.

(ii) Find the range of values of N_2/N for which $T < 0$.

(iii) Find the value of T for which $N_2/N = 1$.

If you cannot solve this problem in terms of a microcanonical ensemble, you can get partial credit and be in a position to answer part c if you use some other method, such as assuming a canonical ensemble.

(5) (c) Derive an expression for U in terms of T and show that the heat capacity dU/dT is positive for all values of T, which insures thermodynamic stability.

33-#1

a.) Has the ship been hit or not? If it has been hit, find the point where the laser beam hits. If the beam fails to hit, calculate the distance by which it misses. You may assume that the two ships pass so close together that propagation time of the laser beam can be neglected

b.) Explain carefully the origin of the apparent paradox. Sketch the relative positions of the two ships ~~at~~ as seen by S' at the moment of firing. Give the coordinates of the fore and aft of each ship in terms of some convenient set of axes

Solution

Denote event E_1, the coincidence of the front of S with the tail of S'. Assume that the origins of their respective coordinate systems coincide at this point and time so that $X_1 = X_1' = 0$ and $t_1 = t_1' = 0$. Event E_2 corresponds the ~~th~~ gun firing. E_1 and E_2 are simultaneous in S, so $t_2 = 0$.

In the primed system

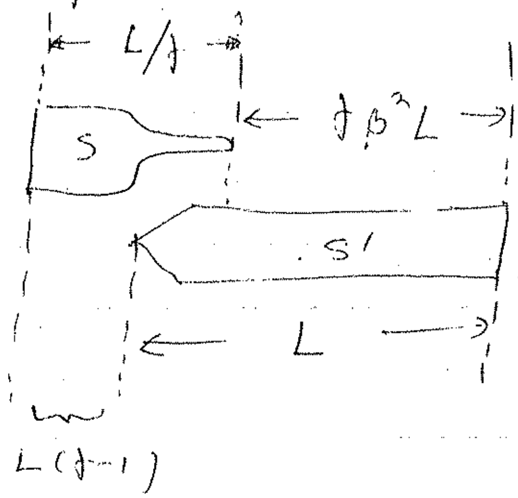
$$X_2' = \gamma(X_2 - u t_2) = \gamma L$$

Consequently the shot misses by a distance $L(\gamma - 1)$.

The difference is due to relativity of simultaneity. ~~Since~~ E_1 and E_2 are simultaneous in S, but since S' is moving toward E_2, E_2' seems to occur before E_1'. In fact

$$t_2' = \gamma\left(t_2 - \frac{u}{c^2} X_2\right) = -\frac{\gamma u}{c^2} L$$

ie. the tail of S' is a distance $\frac{\gamma u^2 L}{c^2}$ in front of S at the moment of firing. the correct picture is



33-2 QM Solution

a.) $J_\pm |j,m\rangle = \hbar \sqrt{(j\mp m)(j\pm m+1)} \; |j, m\pm 1\rangle$

Assume $\psi_{1/2} = A\, \ell_1 \chi_{-1/2} + B\, \ell_0 \chi_{1/2}$

Use $L_\pm$ and $S_\pm$ as the raising + lowering operators operating on the $j=1$ and $j=1/2$ subspaces.

$J_+ \psi_{1/2} = (S_+ + L_+)\psi_{1/2} = 0 = A\, \ell_1 \chi_{1/2} + \sqrt{2} B\, \ell_1 \chi_{1/2}$

so $A = -\sqrt{2} B$

Normalizing:

$|\psi_{1/2}|^2 = |A|^2 (1 + 1/2) = \frac{3}{2}|A|^2 = 1$

so $\boxed{\psi_{1/2} = \sqrt{\frac{2}{3}}\, \ell_1 \chi_{-1/2} - \sqrt{\frac{1}{3}}\, \chi_{1/2} \ell_0}$

now

$\psi_{-1/2} = J_- \psi_{1/2} = \hbar \psi_{-1/2} = \sqrt{\frac{2}{3}} \cdot \sqrt{2}\, \ell_0 \chi_{-1/2}$

$- \sqrt{\frac{1}{3}}\, \ell_0 \chi_{-1/2} - \sqrt{\frac{1}{3}} \cdot \sqrt{2}\, \chi_{1/2} \ell_{-1}$

or $\boxed{\psi_{-1/2} = \frac{1}{\sqrt{3}}\, \ell_0 \chi_{-1/2} - \sqrt{\frac{2}{3}}\, \ell_{-1} \chi_{1/2}}$

b) $Y_{10} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \qquad Y_{1\pm1} = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{\pm i\phi}$

$|\psi_{1/2}|^2 \propto \left[\sqrt{2}\, \ell_1 \chi_{-1/2} - \ell_0 \chi_{1/2}\right]^2$

$\propto 2|\ell_1|^2 - 2\sqrt{2}\,(\ell_1 \chi_{-1/2}, \ell_0 \chi_{+1/2}) + |\ell_0|^2$

$= \left(\frac{3}{4\pi}\right) \sin^2\theta + \left(\frac{3}{4\pi}\right) \cos^2\theta = \text{const.}$

and similarly for $|\psi_{-1/2}|^2$.

#3 : Solutions

(a) the wavelike properties of electrons satisfying $p = \hbar\, 2\pi/k$

(b) i) Frequency of light (or wavelength)
ii) work function of metal (includes surface too)
iii) Retarding potential on metal
Contact potential
Direction of incident light
Charge on e
temperature

iv) intensity of light.

Evidence that light is composed of photons (particles) each with energy $= \hbar \nu$.

(c) Why the heat capacity curve had the shape 

,
i.e. why low Temperature showed quantum effect wit $C_v \propto T^3$, whereas high temperature showed Classical result. (Einstein assumed all multiples of one characteristic frequency; deBye limited his maximum ν, but permitted a distribution of ν's).

(d) (i) $1/2\ \mathring{A} = .5 \cdot 10^{-8}$ cm
(ii) 2×10^{-8} cm
(iii) $.7\ \text{fm} = .7 \cdot 10^{-13}$ cm
(iv) $6\ \text{fm} = 6 \cdot 10^{-13}$ cm
(V) $\sim 10^{10}$ lightyears $= 10^{10} \cdot 3 \cdot 10^{10}\ \text{cm/sec} \times \pi \cdot 10^{7}\ \text{sec/year} \simeq 10^{28}$ cm

Soltn. #4

$H_h = \mu_0 \underset{\sim}{L}\cdot\underset{\sim}{B}$, $H = \frac{p_r^2}{2\mu} + \frac{L^2}{2mr^2} + \mu_0 \underset{\sim}{L}\cdot\underset{\sim}{B}$

$\frac{p_r^2}{2\mu} = 0$ for rigid rotor

ⓐ

$\underset{\sim}{B} = B_x \hat{e}_x$

$$H = \frac{L_x^2 + L_y^2 + L_z^2}{2mr^2} + \mu_0 L_x B = \frac{L^2}{2I_0} + \mu_0 L_x B$$

a i) C = Const of motion $\Rightarrow$ $[\mathcal{H}, C] = 0$

$[L_x, L_x] = 0$, $[L_x, L_{y,z}] \neq 0$ $\therefore$ L_x = Const, L_y, L_z not

$[L^2, L^2] = [L^2, L_x] = 0$ — L^2 constant

$P\mathcal{H} = \mathcal{H}P$ (L_x, B are pseudo vector) — P conserved

(ii) $E = \frac{\ell(\ell+1)\hbar^2}{2I_0} + \mu_0 B m \hbar$ (m = x projection)

ℓ = integer

B ≠ 0 $-\ell \le m \le \ell$ { $2\ell+1$ values non degenerate

B = 0 $2\ell+1$ degeneracy

(b) (i) $\Psi(t=0) = \frac{1}{4}\sqrt{3/\pi}\,(1+\cos\theta)$

this is superposition of 2 eigenstates

$Y_{00} = 1/\sqrt{4\pi}$ $Y_{10} = \sqrt{3/4\pi}\ P_1 = \sqrt{3/4\pi}\cdot\cos\theta$

$\Psi = \sqrt{3/4}\, Y_{00} + \sqrt{\frac{1}{4}}\, Y_{10}$

Prob = 3/4 : $\ell m = 00$
1/4 $\ell m = 10$

B=0

(ii) $\Psi_0 = \sqrt{3/4}\,|00\rangle + \sqrt{1/4}\,|10\rangle$, $\Psi(t) = \sqrt{3/4}\,|00\rangle e^{-iE_0t/\hbar} + \sqrt{1/4}\,|10\rangle e^{-iE_1t/\hbar}$

$= \sqrt{3/4}\, Y_{00}\, e^{-it\cdot 0} + \sqrt{1/4}\, Y_{10}\, e^{-i\hbar t/I_0}$

(1)

5 a) Starting at the lowest energy, each eigenstate has one more node. ψ_0 has zero nodes, so there are two eigenstates with energies less than E_1

b) Each eigenstate has an energy higher than the next lower one given approximately by

$$\Delta E = \frac{\Delta p^2}{2m}$$

where

$$\Delta p = \frac{\hbar}{a}$$

so

$$\Delta E = \frac{\hbar^2}{2ma^2}$$

The energy interval $V_0 - E_1$ can therefore accomodate approximately

$$N = \frac{V_0 - E_1}{\left(\frac{\hbar^2}{2ma^2}\right)} \quad \text{eigenstate}$$

c)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

For $|x| \le \frac{a}{2}$ $\quad \psi = \text{const}\begin{cases}\sin\alpha x \\ \cos\alpha x\end{cases}$ $\quad$ where $\alpha = \sqrt{\frac{2mE_1}{\hbar^2}}$

ψ is even, so

$$\frac{\rho(\frac{a}{2})}{\rho(0)} = \frac{|\psi(a/2)|^2}{|\psi(0)|^2} = \cos^2\left[\sqrt{\frac{2mE_1}{\hbar^2}}\,\frac{a}{2}\right]$$

c) at $x \ge \frac{a}{2}$ $\quad \psi_2 = c_2 e^{-\alpha' x}$ $\quad \alpha' = \sqrt{\frac{2m(V_0-E_1)}{\hbar^2}}$

$x \le a/2$ $\quad \psi_1 = c_1 \cos\alpha x$

at $x = \frac{a}{2}$

$$\psi_1 = \psi_2 \qquad c_2 e^{-\alpha'\frac{a}{2}} = c_1 \cos\alpha\frac{a}{2}$$

$$c_2 = c_1 \cos\left(\frac{\alpha a}{2}\right) e^{\alpha' a/2}$$

$$\frac{\rho(a)}{\rho(0)} = \left(\frac{c_2}{c_1}\right)^2 e^{-2\alpha' a} = \cos^2\left(\frac{\alpha a}{2}\right) e^{-\alpha' a}$$

$$= \cos^2\left[\frac{a}{2}\sqrt{\frac{2mE_1}{\hbar^2}}\right]\exp\left[-\frac{a}{\hbar}\sqrt{2m(V_0-E_1)}\right]$$

6 a)

The Second law is $\delta S \geq 0$ for a spontaneous process. If ~~an a~~ system 1 at temperature T_1 and energy E_1 is placed in thermal contact with system 2 at temperature T_2 and energy E_2, then if $T_1 < 0$ and $T_2 > 0$, we have from the second law

$$\delta S = \delta S_1 + \delta S_2 \geq 0$$

By the first law

~~$\delta E_1 = T\delta S_1 + \delta E_2 - T\delta$~~

$$\delta E_1 + \delta E_2 = 0$$

But $\delta E_1 = T_1 \delta S_1$ and $\delta E_2 = T_2 \delta S_2$

so that

$$\delta S_1 = \frac{\delta E_1}{T_1} \qquad \delta S_2 = \frac{\delta E_2}{T_2} = -\frac{\delta E_1}{T_2}$$

$$\delta S = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\delta E_1 \geq 0$$

If $T_1 < 0$ and $T_2 > 0$, $\frac{1}{T_1} - \frac{1}{T_2} < 0$ so that $\delta E_1 < 0$

Thus heat always flows from a system at negative temperature to a system at positive temperature.

6 8) In a microcanonical ensemble

$$U = n_1 E_1 + n_2 E_2 \qquad E_1 = -\epsilon$$

$$\frac{U}{N} = -(1-f)\epsilon + f\epsilon \qquad E_2 = +\epsilon$$

$$U = N\epsilon\,(2f-1) \qquad f = \frac{n_2}{N}$$

$$S = k \ln \frac{N!}{N_1!\,N_2!} = -kN_1 \ln N_1 - kN_2 \ln N_2 + kN \ln N$$

$$= -kN(1-f)\ln(1-f)N - kfN\ln fN + kN\ln N$$

$$= -kN(1-f)\ln(1-f) - kNf\ln f$$

$$T = \frac{\partial U}{\partial S} = \frac{dU/df}{dS/df}$$

$$\frac{dU}{df} = 2\epsilon N$$

$$\frac{dS}{df} = kN\ln(1-f) - kN\ln f = kN\ln\frac{1-f}{f}$$

$$T = \frac{2\epsilon N}{kN\ln\frac{1-f}{f}} = \frac{2\epsilon}{k\ln\left(\frac{1-f}{f}\right)}$$

If $f > \frac{1}{2}$, $T < 0$

$f = 1$ $T = -0$

6 c)

$$\frac{1-f}{f} = \exp\left(\frac{2\epsilon}{kT}\right)$$

$$f = \frac{1}{1+e^{\frac{2\epsilon}{kT}}} = \frac{e^{-\epsilon/kT}}{e^{-\epsilon/kT} + e^{+\epsilon/kT}}$$

$$U = \epsilon N(2f-1) = -2\epsilon N \tanh\frac{\epsilon}{kT}$$

$$C = \frac{dU}{dT} = +\frac{2\epsilon^2 N}{kT^2}\operatorname{sech}^2\frac{\epsilon}{kT}$$

~~Since~~ $C > 0$ for all values of T