General Instructions

This Comprehensive Examination for Winter 1979 (#32) consists of six problems of equal weight (20 points each). Half of the material is judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show as clearly as possible all your steps so that partial credit can be given liberally in case you do not complete a problem or make errors. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

 Possibly useful formulae:

\[ E_n = E_n + \langle n|V|n\rangle + \sum_{m \neq n} \frac{|\langle n|V|m\rangle|^2}{E_n - E_m} \]

\[ U = \sum_{n=1} A_n r^n e^{in\theta} + \sum_{n=0} B_n r^n e^{in\theta} + A_0 \ln r \]

\[ U = \sum_{n=0} A_n r^n p_n (\cos \theta) + \sum_{n=0} B_n r^{-(n+1)} p_n (\cos \theta) \]

\[ \frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^\infty p_k(z) t^k \]

\[ \mathbf{B} = \frac{\mathbf{\mu}_0}{4\pi} \left[ \frac{\mathbf{r}}{r^3} + \frac{3(\mathbf{r} \cdot \mathbf{r}) \mathbf{r}}{r^5} \right] \]

\[ \mathbf{B} = \frac{\mathbf{\mu}_0}{4\pi} \int \mathbf{\nabla} \times \left[ \frac{\mathbf{M}(r-r')}{|r-r'|^3} \right] \, dv' \]

\[ \mathbf{B} = \frac{\mathbf{\mu}_0}{2\pi} \mathbf{\kappa} \]
1. The vibrations in a diatomic molecule may be represented by a Hamiltonian of the form

\[ H_{\text{vib}} = \hbar \omega (N + 1/2) + E_0 \]

where \( N = 0,1,2,\ldots \). Assume we have a single atom at a temperature \( T \), and that this vibrational motion is completely independent of all other motions.

(a) What is the value of the probability that the atom will be in a state with \( N = n \)?

(b) Sketch (roughly) the probability in part (a) as a function of \( \hbar \omega / kT \) for \( n = 0 \) and for \( n = 5 \).

(c) What is the value of \( \langle H_{\text{vib}} \rangle \), where \( \langle A \rangle \) is the average value of \( A \) at thermal equilibrium?

(d) What is the value of \( \langle [H_{\text{vib}}, H_{\text{vib}}]^2 \rangle \)?

(e) Evaluate your results for parts (c) and (d) in the classical limit.
2. An electromagnetic wave of angular frequency $\omega$ and wave number $k$ is propagating unattenuated in the $x$ direction between perfectly conducting plates lying in the planes $z = 0$ and $z = d$. The space between the plates is a vacuum. The wave is plane polarized, with the electric field vector $E$ in the $y$ direction.

![Diagram of electromagnetic wave](image)

(a) Write down Maxwell's equations for the $B$ and $E$ fields between the plates and the boundary conditions which apply at the plates.

(b) Prove that the $B$ and $E$ vectors are independent of $y$.

(c) Derive a differential equation which determines the dependence of the magnitude of $E$ on $z$.

(d) Solve the equation of (c) and write down explicit solutions in space and time for all non-vanishing components of $\vec{E}$ and $\vec{B}$. 

[Diagram of electromagnetic wave]
3. A dielectric sphere of radius $R$ with dielectric constant $K$ has a permanent polarization $\vec{P}$, which is uniform in direction and magnitude. This gives rise to an electric field $\vec{E}$. Determine the electric field $\vec{E}$ both inside and outside the sphere.

4. A quantum mechanical system that has energy eigenstates is represented by the Hamiltonian matrix

$$
H = \begin{pmatrix}
E_1 & 0 & a \\
0 & E_1 & b \\
a^* & b^* & E_2
\end{pmatrix}
$$

where $a$ and $b$ are "small" in the sense that $|a|, |b| \ll E_1, E_2$.

(a) Use ordinary stationary-state perturbation theory to find the energy eigenvalues to second order.

(b) Find the exact eigenvalues.

(c) Compare the results from (a) and (b) by expanding the exact solution in a power series and keeping the second order terms. Do the two solutions agree? If not, explain carefully what the difficulty is. How would one overcome this problem in a practical calculation?
5. (a) A perfect gas at pressure $P$ and temperature $T$ is in a compartment separated from a vacuum by a thin wall containing a circular aperture of radius $R$, which is small compared to the mean free path $L$ of the gas. Starting with the Maxwell Boltzmann distribution law for the molecules reaching the aperture

$$dn(v) = \text{const} \exp \left[ -\frac{mv^2}{2kT} \right] dv_x dv_y dv_z$$

derive the rate of flow $I$ of the gas through the aperture. Show that it is of the form

$$I = cT^t P^p$$

and evaluate the constants $c$, $t$, and $p$.

(b) Suppose that instead of a vacuum, the other side of the thin wall is a compartment in which there is the same gas held at a temperature $T_b$, which is different from the temperature $T_a$ in the first compartment. Determine the steady-state ratio of pressures in the two compartments $P_b/P_a$.

(c) Estimate the mean free path $L$ for a noble gas like argon using a model for the interaction between the atoms in which the atoms are hard spheres of radius "a" with attractive forces of range slightly larger than "a" (i.e., $\bar{F} \approx 0$ when the separation $r$ between atoms obeys $r - 2a > a$). Estimate the value of "a" from the experimental value of the density $d$ of crystalline argon, which has a face centered lattice with four atoms in a cubic unit cell.
6. (a) A solenoid of radius \( R \) has \( N \) turns wound uniformly over a length \( L \) and a current \( I \). Derive the magnetic field \( B \) on the axis of the solenoid and exterior to it at a distance \( d \) from one end.

(b) A toroid of length \( L \) and cross-section area \( A \) contains a magnetic material with a permeability \( \mu >> \mu_0 \). Assume that \( L >> A^{1/2} \). A coil of \( N \) turns of wire wrapped around the toroid has a current \( I \). There is an air gap in the toroid with a uniform width \( w \), where \( w << L \). Derive an expression for the field in the gap as a function of \( t \), which is accurate to first order in \( w/L \).
The vibrations in a diatomic molecule may be
represented by a Hamiltonian of the form

$$\mathcal{H}_{vb} = \hbar \omega (N + 1/2) \pm \epsilon_0$$

where $N = 0, 1, 2, 3 \ldots \infty$. Assume we have a single atom at
temperature $T$ and that this vibrational motion is
completely independent of all other motions. What extra
value of

(a) What is the probability that the atom will be in a
state with $N = n$?

(b) Sketch (roughly) the probability in (a) as a function
of $\hbar \omega / kT$ for $n = 0$ and $n = 5$.

c) What is $\langle \mathcal{H}_{vb} \rangle$?

d) What is $\langle (\mathcal{H}_{vb} - \langle \mathcal{H}_{vb} \rangle)^2 \rangle$?

(e) What would you do

(f) Evaluate your results for parts (c) and (d) in

the classical limit.
Solution

Partition function
\[ Z = \sum e^{-\beta E} = e^{-\beta (E_0 + kT\omega)} \sum e^{-\beta k\omega} = e^{-\beta E_0} \frac{1}{1 - e^{-\beta k\omega}} \]

(a) \[ P_n = \frac{e^{-\beta \mu n}}{Z} = e^{-\beta \mu n} (1 - e^{-\beta k\omega}) \]

(b) Let \( \chi = \frac{b\omega}{kT} \)
\[ P = e^{-\beta \chi} - e^{-\beta \omega} \]
\[ P_0 = 1 - e^{-\beta \chi}, \quad P_0 - e^{-\beta \chi} \]
\[ \chi \]

(c) \[ \langle \chi \rangle = -\frac{\partial}{\partial \beta} \ln Z = -\frac{1}{\partial \beta} [ \ln (1 - e^{-\beta k\omega}) ] \]
\[ = E_0 + \frac{k\omega e^{-\beta k\omega}}{1 - e^{-\beta k\omega}} = E_0 + \frac{k\omega}{e^{\beta k\omega} - 1} \]

(d) \[ \frac{1}{\partial \beta} \langle \chi \rangle = \langle [\chi - \langle \chi \rangle]^2 \rangle = -\frac{2}{\partial \beta} \left[ E_0 + \frac{k\omega}{e^{\beta k\omega} - 1} \right] \]
\[ = \frac{(k\omega)^2 e^\beta k\omega}{(e^{\beta k\omega} - 1)^2} \]

(3) Except classical limit when high Temp.
\[ \langle \chi \rangle \approx E_0 + \frac{k\omega}{1 - 1 + \frac{k\omega}{kT}} = E_0 + \frac{k\omega}{kT} \]
\[ \approx \frac{1}{\partial \chi} \text{(small)} \]
\[ \langle \chi^2 \rangle = \frac{(k\omega)^2}{(1 - 1 + \frac{k\omega}{kT})^2} \approx \frac{1}{\beta^2} \]
An electromagnetic wave of angular frequency \( \omega \) and wave number \( k \) is propagating unattenuated in the \( z \) direction between two perfectly conducting plates \( \sigma \) lying in the planes \( z = 0 \) and \( z = d \). The space between the plates is vacuum. The wave is plane polarized, with the electric field vector in the direction \( y \).

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(a) Write down Maxwell's equations for the fields between the plates and the boundary conditions which apply at the plates.

(b) Prove that the magnitude of the wave is independent of \( y \).

(c) Derive a differential equation which tells how the magnitude of \( \vec{E} \) depends on \( z \).

(d) Solve the equation of (c) and write down explicit solutions in space and time for all the nonvanishing components of \( \vec{E} \) and \( \vec{H} \).

(e) Find the smallest value of \( \omega \) such that the wave can propagate without attenuation.
(a) $\mathbf{E} \cdot \mathbf{B} = 0$

\[ \mathbf{A} \cdot \mathbf{E} = 0 \]

\[ \mathbf{A} \cdot \mathbf{B} = 0 \]

\[ \begin{align*}
\nE &= E_0 \hat{e}_y e^{(kz - \omega t)} \\
E_{\text{tangent}} &\text{ and } B_{\text{normal}} \text{ vanish at walls } z = 0, d
\end{align*} \]

(b) $\nabla \cdot \mathbf{E} = 0$ \implies $\frac{\partial E_y}{\partial y} = 0$. 

So the only $E$ component is independent of $y$.

For $B$:

\[ \nabla \times \mathbf{B} = -i \omega \mathbf{E} \hat{e}_y \]

\[ \begin{align*}
(1) \text{ x component:} & \quad \frac{\partial B_z}{\partial y} = \frac{2B_y}{2z} \\
(2) \text{ z component:} & \quad \frac{\partial B_x}{\partial z} = \frac{2B_y}{2y} \\
(3) \text{ y component:} & \quad \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c} E
\end{align*} \]

\[ \nabla \times \mathbf{E} = \frac{c}{\omega} \mathbf{B} \implies \nabla \times E = \hat{e}_x \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{e}_y \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + \hat{e}_z \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \]

\[ = -\hat{e}_x \frac{\partial E_y}{\partial z} + \hat{e}_z \frac{2E_y}{2x} \]

(4) $i\omega \mathbf{B} = -\hat{e}_x \frac{\partial E_y}{\partial z} + i \mathbf{e}_z E$

So $B_y = 0$ (: no $y$ variation), using (1) above \( \implies \frac{\partial B_z}{\partial y} = 0 \)

" \( \implies \frac{\partial B_x}{\partial y} = 0 \)"
C 7 differential E4m

\text{eq}(1) \Rightarrow B_x = -\frac{\varepsilon}{\omega} \frac{\partial E_y}{\partial z}, \quad B_z = \frac{\varepsilon k}{\omega} E, \quad B_y = 0

(3) \frac{\partial^2 B_y}{\partial z^2} - \frac{\partial^2 B_z}{\partial x^2} = -\frac{\varepsilon}{\omega} \frac{\partial E}{\partial z} - \frac{c k}{\omega} \frac{\partial E}{\partial x} = -\frac{\varepsilon}{c} \frac{\partial E}{\partial z}

+ \frac{c}{\omega} \frac{\partial^2 E}{\partial z^2} - \frac{c k}{\omega} \frac{\partial^2 E}{\partial x^2} = -\frac{\varepsilon}{c} \frac{\partial E}{\partial z}

\frac{\partial^2 E}{\partial z^2} - k^2 E = -\frac{\omega^2}{c^2} E

\begin{cases}
\frac{\partial^2 E}{\partial z^2} = (k^2 - \omega^2/c^2) E = -(\omega^2/c^2 - k^2) E
\end{cases}

E_y = 0, \quad at \ z = 0, \ d

\text{(b)} \text{E}(z) = A \sin k z + B \cos k z

E(0) = 0 \Rightarrow B = 0
E(d) = 0 \Rightarrow \int_{0}^{d} k dz = \frac{\pi}{n}

\quad \Rightarrow \quad k = \frac{\pi n}{d} = \sqrt{\frac{\omega^2 c^2 - k^2}{\omega^2 c^2}}

\bar{E} = E_0 \sin \left( \frac{n \pi z}{d} \right) e^{i(kx - \omega t)} e_y

\begin{cases}
B_y = 0, \quad B_z = \frac{c k}{\omega} E = \frac{c k}{\omega} E_0 \sin \left( \frac{n \pi z}{d} \right) e^{i(kx - \omega t)}
\end{cases}

\begin{cases}
B_y = -\frac{\varepsilon}{\omega} \frac{\partial E_y}{\partial x} = -\frac{\varepsilon}{\omega} E_0 \frac{\pi n}{d} \cos \left( \frac{n \pi z}{d} \right) e^{i(kx - \omega t)}
\end{cases}
A dielectric sphere of radius $R$ has a permanent polarization $P^p$; which is uniform in direction and magnitude. The polarized sphere gives rise to an electric field. Determine this field both inside and outside the sphere.

Solution: Since $\nabla \cdot \vec{P}^p = 0$, there are no bound charges except at the surface of the sphere. Hence $\nabla \cdot \vec{E} = 0$ everywhere except at the surface. The simplest relevant solutions to Laplace's equation are

- $V_{\text{outside}} = AR^{-2} \cos \theta$ (Dipole potential)
- $V_{\text{inside}} = BR \cos \theta$ (Constant $E$)

The constants $A, B$ are found from the boundary conditions at the surface of the sphere.

a.) Tangential components of $\vec{E}$ are equal:

$$E_{\text{tang}} = E_\theta = -\frac{1}{b^2} \frac{\partial}{\partial \theta} (\frac{1}{\sin \theta} U)$$

$$E_\theta = AR^{-3} \sin \theta = B \sin \theta = E_\theta$$

so $A = BR^3$

b.) The normal comp. of $\vec{D}$ are equal:

$$D_{\text{norm}} = D_r = \varepsilon_0 E_r + P_r = -\varepsilon_0 \frac{\partial E_\theta}{\partial \theta} + P \cos \theta$$

$$D_{\text{norm}} = 2AR^{-2} \cos \theta = -B \varepsilon_0 \cos \theta + P \cos \theta = D_{\text{norm}}$$

$$3 \varepsilon_0 B = P$$

$$B = \frac{P}{3 \varepsilon_0}$$

$$A = \frac{PR^3}{3 \varepsilon_0}$$
The fields are:

\[ E_{\theta}^{\text{out}} = \frac{PR^3}{3\varepsilon_0 r^3} \sin \theta \]
\[ E_{\theta}^{\text{in}} = \frac{PR^3}{3\varepsilon_0} \sin \theta \]
\[ E_r^{\text{out}} = \frac{\theta PR^3}{3\varepsilon_0 r^3} \cos \theta \]
\[ E_r^{\text{in}} = -\frac{P}{3\varepsilon_0} \cos \theta \]

Note that \[ E_{\theta}^{\text{in}} = -\frac{1}{3\varepsilon_0} P \] is the "dipole field."
A quantum mechanical system that has three energy eigenstates is represented by the Hamiltonian matrix

$$H = 
\begin{pmatrix} 
E_1 & a & c \\
\bar{c} & E_1 & b \\
a^* & b^* & E_2 
\end{pmatrix}$$

where $a$ and $b$ are "small" in the sense that $|a|, |b| < E_1, E_2.$

1. Use ordinary stationary state perturbation theory to find the energy eigenvalues to second order.

2. Find the exact eigenvalues by diagonalizing the matrix.

3. Compare the results from 1) and 2) by expanding the exact solution in a power series and keeping the second order terms. Are the two solutions agreed? If not, explain carefully what the difficulty is. How would one overcome this problem in a practical calculation?
4.) Let \( H = H_0 + H' \) where \( H' = \begin{pmatrix} C & c & q \\ C & 0 & b \\ 0 & b^* & \alpha + b^* c \end{pmatrix} \)

is the perturbation. In second order,

\[
W_m = E_m + H_{mm} \sum_{n \neq m} \frac{1}{E_m - E_n} \]

where \( H_{mm} = W_m H_m \) and \( H_0 u_m = E_m u_m \).

\[
W_1 = E_1 + \frac{|a|^2}{E_1 - E_2} \quad W_2 = E_1 + \frac{|b|^2}{E_1 - E_2} \quad W_3 = E_2 + \frac{|a|^2 + |b|^2}{E_2 - E_1}
\]

2.) The exact eigenvalues are solutions of the equation

\[
\begin{pmatrix}
E_1 - w & a & 0 \\
0 & E_1 - w & b \\
a^* & b^* & E_2 - w
\end{pmatrix} = (E_1 - w) \left[ (E_1 - w)(E_2 - w) - |b|^2 / |a|^2 \right]
\]

\[
W_1 = E_1 \quad W_2, W_3 = \frac{E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4(|a|^2 + |b|^2)}}{2}
\]

3.) To second order:

\[
W_2 = E_1 + \frac{|a|^2 + |b|^2}{E_1 - E_2} \quad W_3 = E_2 + \frac{|a|^2 + |b|^2}{E_2 - E_1}
\]

The two solutions fail to agree because two of the unperturbed eigenfunctions are degenerate. It is not known a priori what linear combination of these eigenfunctions the exact solution collapses into when the perturbation is removed, hence the named expansion fails. In a practical calculation one would have to diagonalize a submatrix sufficiently large to contain enough of the perturbation to
7.5

a) The flux per unit area is

\[ J = \text{const} \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-\frac{1}{2} \frac{m}{kT} \left( x^2 + y^2 + z^2 \right)}{r_x r_y r_z} \, dx \, dy \, dz \]

The molecular density is

\[ \frac{N}{V} = \text{const} \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-\frac{1}{2} \frac{m}{kT} \left( x^2 + y^2 + z^2 \right)}{r_x r_y r_z} \, dx \, dy \, dz \]

\[ \frac{J}{N/V} = \frac{\int_0^\infty N_x \, \frac{-\frac{1}{2} m r_x^2 / kT}{\sqrt{2 \pi m r_x^2 / kT}} \, dx}{\int_{-\infty}^{\infty} \frac{-\frac{1}{2} m r_x^2 / kT}{\sqrt{2 \pi m r_x^2 / kT}} \, dx} = \sqrt{\frac{2kT}{m}} \int_0^\infty \frac{-y^2}{\sqrt{2\pi y^2}} \, dy \]

\[ = \frac{1}{2} \sqrt{\frac{2kT}{m}} \sqrt{\pi} \]

By the ideal gas law \( \frac{N}{V} = \frac{p}{kT} \)

The total particle flow is \( I = J \pi R^2 \)

\[ I = \pi R^2 J = \pi R^2 \frac{p}{kT} \sqrt{\frac{kT}{2\pi m}} = \frac{\pi R^2 p}{\sqrt{2\pi m kT}} \]

\[ c = \frac{\pi R^2}{\sqrt{2\pi m kT}} \]

\( k = -\frac{1}{2} \) \( \mu = 1 \)
(b) At a steady state

\[ J_{a \rightarrow b} = c T_a^{-\frac{1}{2}} P_a \]

\[ J_{b \rightarrow a} = c T_b^{-\frac{1}{2}} P_b \]

\[ J_{a \rightarrow b} = J_{b \rightarrow a} \]

\[ T_a^{-\frac{1}{2}} P_a = T_b^{-\frac{1}{2}} P_b \]

\[ \frac{P_a}{P_b} = \left( \frac{T_a}{T_b} \right)^{\frac{1}{2}} \]

(c) For a spherical atom, we can derive:

\[ L = \pi (2a)^2 = \frac{V}{N} = \frac{kT}{p} \]

\[ L = \frac{kT \cdot \sigma}{4p \pi a^2} \]

For a FCC crystal:

\[ d = \frac{m}{V} = \frac{4M}{N_{av} \cdot \frac{L^3}{2}} \]

\[ \ell \text{ is related to } a \text{ by} \]

\[ 2a = \frac{4L}{\sqrt{2}} \]

\[ L = 2^{\frac{3}{2}} a \]

\[ d = \frac{4M}{N_{av} \cdot 8a^3} \]

\[ a^2 = \frac{4\pi (M/2N_{av})^{\frac{2}{3}}}{\frac{4\pi}{3}} \]

\[ L = \frac{kT}{4\pi p} \left( \frac{M}{2N_{av}} \right)^{\frac{2}{3}} \]
\[ dB = \frac{\mu_0 \mathbf{J} \times \hat{n}}{4\pi r^3} \, dr. \]

The contribution of a length \( dx \) of the solenoid is (in cylindrical coordinates):

\[
\frac{dB}{dz} = \mu_0 \int_0^{2\pi} Rd\theta \left[ \frac{I}{\theta L} \right] \hat{a}_\theta \times \hat{n} \frac{R}{4\pi r^3}
\]

\[
\int_0^{2\pi} \hat{a}_\theta \times \hat{n} \, d\theta = 2\pi R \hat{a}_z
\]

since the contribution from the component of \( \hat{n} \parallel z \) cancels out.

\[
\frac{dB}{dz} = \frac{\mu_0 R^2 I N}{2L} \frac{dz}{r^3}
\]

\[ R = n \sin \theta \quad R = -z \tan \theta \]

\[ n = \frac{R}{\sin \theta} \quad z = -\frac{R}{\tan \theta} \]

\[ d\theta = \frac{R}{\sin^2 \theta} \, d\theta \quad \frac{R}{dz} \]

\[ B = \frac{\mu_0 R^2 I N}{2L} \left[ \tan^{-1} \frac{R}{d + L} \right] \frac{R}{\sin^2 \theta} \frac{dz}{r^3}
\]

\[ = \frac{\mu_0 I N}{2L} \left( \tan^{-1} \frac{R}{dz} \right) \left[ \tan^{-1} \frac{R}{dz} \right] = \frac{\mu_0 I N}{2L} \left[ \frac{d + L}{\sqrt{(d + L)^2 + R^2}} - \frac{d}{\sqrt{d^2 + R^2}} \right] \]
By Ampere law,

\[ \oint C \mathbf{H} \cdot d\mathbf{l} = NI \]

\[ \mathbf{H} = H_a \mathbf{w} + H_x (L-w) \]

\[ H_a = \frac{B_a}{\mu_0} \quad H_x = \frac{B_x}{\mu} \]

\[ B_a = B_x \quad \text{since} \quad m \cdot \mathbf{B} \quad \text{is constant across a surface of discontinuity} \]

\[ NI = B \left( \frac{w}{\mu_0} + \frac{L-w}{\mu} \right) \]

\[ B = \frac{NI}{L} \frac{\mu NI / L}{1 + \omega^2 \left( \frac{\mu}{\mu_0} - 1 \right)} \]