PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #32

January 6, 1979

General Instructions

This Comprehensive Examination for Winter 1979 (#32) consists of six problems of equal weight (20 points each). Half of the material is judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show as clearly as possible all your steps so that partial credit can be given liberally in case you do not complete a problem or make errors. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

Possibly useful formulae:

$$E_n' = E_n + \langle n|V|n\rangle + \sum_{m\neq n} \frac{|\langle n|V|m\rangle|^2}{E_n - E_m}$$

$$U = \sum_{n=1} A_n r^{-n} e^{in\theta} + \sum_{n=0} B_n r^n e^{in\theta} + A_0 \ln r$$

$$U = \sum_{n=0} A_n r^n P_n(\cos\theta) + \sum_{n=0} B_n r^{-(n+1)} P_n(\cos\theta)$$

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{\ell=0} P_\ell(z) t^\ell$$

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{-\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right] \qquad \vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left[\frac{\vec{M}\times(\vec{r}-\vec{r}')}{|r-r'|^3}\right] dv'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times(\vec{r}'-\vec{r})}{|r'-r|^3}\, dv' \qquad \vec{B} = \frac{\mu_0 I}{2\pi a}\vec{k}$$

1. The vibrations in a diatomic molecule may be represented by a Hamiltonian of the form

$$H_{vib} = \hbar\omega\,(N + 1/2) + E_0$$

where $N = 0,1,2,\ldots\infty$. Assume we have a single atom at a temperature T, and that this vibrational motion is completely independent of all other motions.

(a) What is the value of the probability that the atom will be in a state with $N = n$?

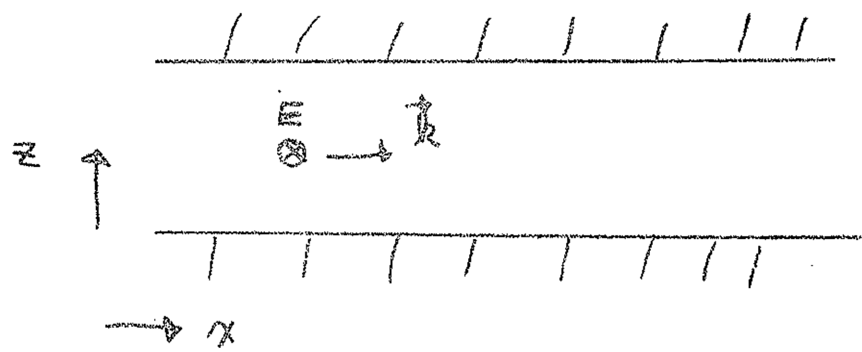
(b) Sketch (roughly) the probability in part (a) as a function of $\hbar\omega/kT$ for $n = 0$ and for $n = 5$.

(c) What is the value of $\langle H_{vib}\rangle$, where $\langle A\rangle$ is the average value of A at thermal equilibrium?

(d) What is the value of $\langle[H_{vib} - \langle H_{vib}\rangle]^2\rangle$?

(e) Evaluate your results for parts (c) and (d) in the classical limit.

2. An electromagnetic wave of angular frequency ω and wave number k is propagating unattenuated in the x direction between perfectly conducting plates lying in the planes $z = 0$ and $z = d$. The space between the plates is a vacuum. The wave is plane polarized, with the electric field vector E in the y direction.



(a) Write down Maxwell's equations for the B and E fields between the plates and the boundary conditions which apply at the plates.

(b) Prove that the B and E vectors are independent of y.

(c) Derive a differential equation which determines the dependence of the magnitude of E on z.

(d) Solve the equation of (c) and write down explicit solutions in space and time for all non-vanishing components of $\vec{E}$ and $\vec{B}$.

3. A dielectric sphere of radius R with dielectric constant K has a permanent polarization $\vec{P}$, which is uniform in direction and magnitude. This gives rise to an electric field $\vec{E}$. Determine the electric field $\vec{E}$ both inside and outside the sphere.

4. A quantum mechanical system that has energy eigenstates is represented by the Hamiltonian matrix

$$H = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}$$

where a and b are "small" in the sense that $|a|, |b| << E_1, E_2$.

(a) Use ordinary stationary-state perturbation theory to find the energy eigenvalues to second order.

(b) Find the exact eigenvalues.

(c) Compare the results from (a) and (b) by expanding the exact solution in a power series and keeping the second order terms. Do the two solutions agree? If not, explain carefully what the difficulty is. How would one overcome this problem in a practical calculation?

5. (a) A perfect gas at pressure P and temperature T is in a compartment separated from a vacuum by a thin wall containing a circular aperture of radius R, which is small compared to the mean free path L of the gas. Starting with the Maxwell Boltzmann distribution law for the molecules reaching the aperture

$$dn(\vec{v}) = \text{const} \exp [-mv^2/2kT] \, dv_x dv_y dv_z$$

derive the rate of flow I of the gas through the aperture. Show that it is of the form

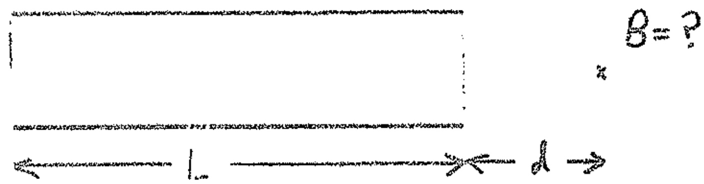
$$I = cT^t p^p$$

and evaluate the constants c, t, and p.

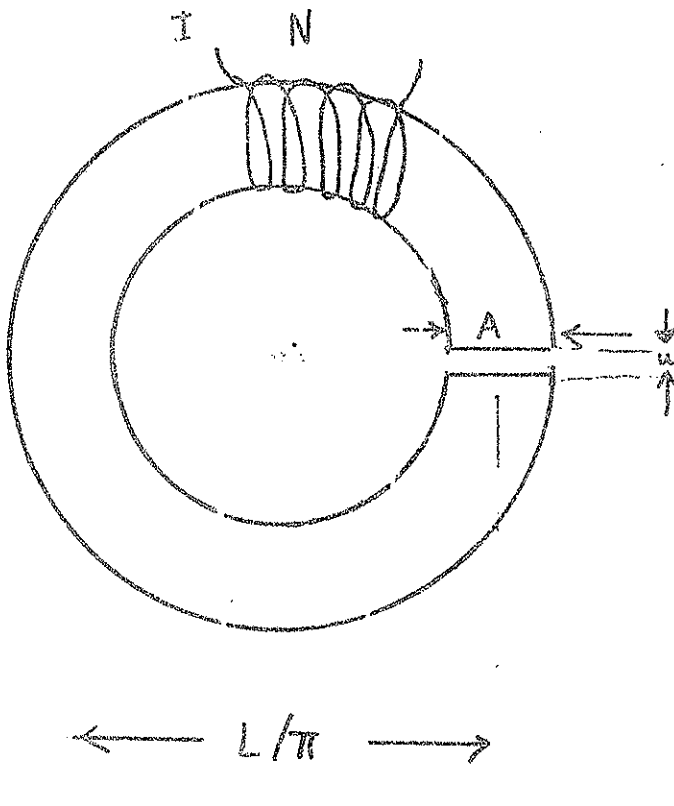
(b) Suppose that instead of a vacuum, the other side of the thin wall is a compartment in which there is the same gas held at a temperature T_b, which is different from the temperature T_a in the first compartment. Determine the steady-state ratio of pressures in the two compartments P_b/P_a.

(c) Estimate the mean free path L for a noble gas like argon using a model for the interaction between the atoms in which the atoms are hard spheres of radius "a" with attractive forces of range slightly larger than "a" (i.e., $\vec{F} \cong 0$ when the separation r between atoms obeys $r - 2a \gtrsim a$). Estimate the value of "a" from the experimental value of the density "d" of crystalline argon, which has a face centered lattice with four atoms in a cubic unit cell.

6. (a) A solenoid of radius R has N turns wound uniformly over a length L and a current I. Derive the magnetic field B on the axis of the solenoid and exterior to it at a distance d from one end.



(b) A toroid of length L and cross-section area A contains a magnetic material with a permeability $\mu \gg \mu_0$. Assume that $L \gg A^{1/2}$. A coil of N turns of wire wrapped around the toroid has a current I. There is an air gap in the toroid with a uniform width w, where $w \ll L$. Derive an expression for the B field in the gap as a function of t which is accurate t) first order in w/L.



#-1

The vibrations in a diatomic molecule may be represented by a Hamiltonian of the form

$$\mathcal{H}_{vib} = \hbar\omega(N+1/2) + E_0$$

where $N = 0, 1, 2, 3 \ldots \infty$. Assume we have a single atom at temperature T and that this vibrational motion is completely independent of all other motions. ~~What is the value of~~

What is the value of

A) the probability that the atom will be in a state with $N = n$.

B) Sketch (roughly) the probability in A as a function of $\hbar\omega/kT$ for $n=0$ and $n=5$

C) What is $\langle \mathcal{H}_{vib} \rangle$?

D) What is $\langle [\mathcal{H}_{vib} - \langle \mathcal{H}_{vib} \rangle]^2 \rangle$

~~E) What would you ... What would you ...~~

E) Evaluate your results for parts (C) and (D) in the classical limit.

Solution

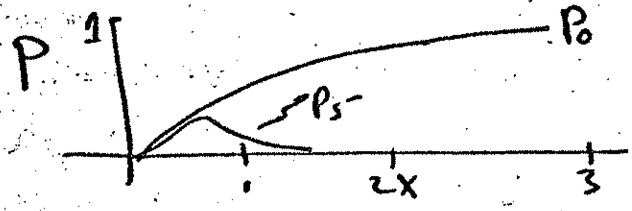
Partition function

$$Z = \sum e^{-\beta H} = \underbrace{e^{-\beta(E_0 + 1/2\hbar\omega)}}_{E'} \sum_N e^{-N\hbar\omega} = e^{-\beta E'} \frac{1}{1-e^{-\beta\hbar\omega}}$$

(A) $$P_n = \frac{e^{-\beta H_n}}{Z} = e^{-\beta\hbar\omega n}(1-e^{-\beta\hbar\omega})$$

(B) let $x = \hbar\omega/kT$, $P_n = e^{-nx} - e^{-(n+1)x}$

$P_0 = 1-e^{-x}$, $P_5 = e^{-5x} - e^{-6x}$



(C) $$\langle H \rangle = -\partial/\partial\beta \log Z = -\frac{\partial}{\partial\beta}\left[-\beta E' - \log(1-e^{-\beta\hbar\omega})\right]$$

$$= E' + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = E' + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

D) $$-\frac{\partial}{\partial\beta}\langle H \rangle = \langle [H - \langle H \rangle]^2 \rangle = -\partial/\partial\beta \left[E' + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right]$$

$$= \frac{(\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}$$

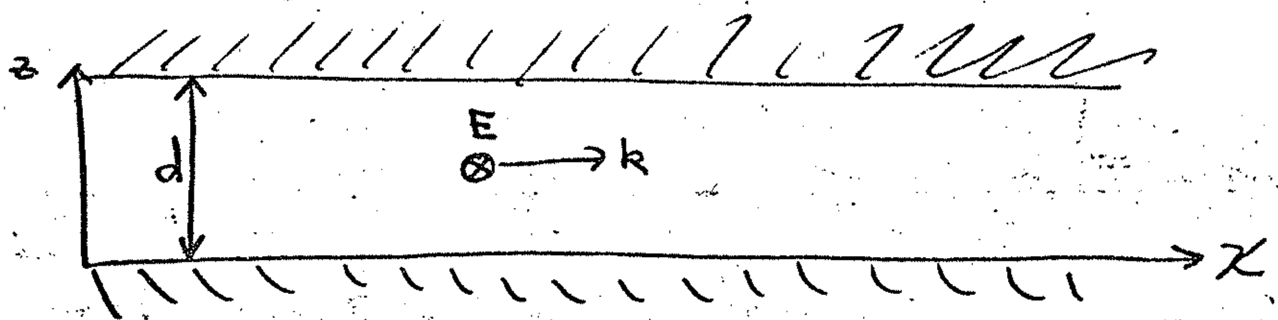
E) Except classical limit when high Temp

$$\langle H \rangle \simeq E' + \frac{\hbar\omega}{1-1+\frac{\hbar\omega}{kT}} = \underbrace{E_0 + \tfrac{1}{2}\hbar\omega}_{\text{zero point (small)}} + 1/\beta$$

$$\langle \delta H^2 \rangle = \frac{(\hbar\omega)^2}{(1-1+\frac{\hbar\omega}{kT})^2} \simeq \frac{1}{\beta^2}$$

#2

An electromagnetic wave of angular frequency ω and wave number k is propagating unattenuated in the x direction between two perfectly conducting plates ~~a~~ lying in the planes $z=0$ and $z=d$. The space between the plates is vacuum. The wave is plane polarized, with its electric field vector in the y direction



(a) Write down Maxwell's equations for the fields between the plates and the boundary conditions which apply at the plates.

(b) Prove that the B and E vectors ~~magnitude of the~~ wave are ~~is~~ independent of y.

(c) Derive a differential equation which tells how the magnitude of ~~the~~ ~~wave~~ E depends on z.

(d) Solve the equation of (c) and write down explicit solutions in space and time for all the nonvanishing components of $\vec{E}$ and $\vec{H}$.

(e) Find the smallest value of ω ~~such that the~~ wave can propagate without attenuation.

Solution

(a) $\nabla \cdot B = 0$ $\qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = i\frac{\omega}{c}\vec{B}$

$\nabla \cdot E = 0$ $\qquad \nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} = -i\frac{\omega}{c}\vec{E}$

$$E = E_0\, \hat{e}_y\, e^{i(kx-\omega t)}$$

$E_{Tangent}$ & B_{normal} vanish at walls $z = 0, d$

(b) $\nabla \cdot E = 0 \Rightarrow \frac{\partial E_y}{\partial y} = 0$ } so the only E component is indep of y.

now for B

$\nabla \times B = -i\frac{\omega}{c} E\,\hat{e}_y \Longrightarrow$ (1) x component
$\partial B_z/\partial y = \partial B_y/\partial z$

(2) z component
$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$

(3) y component
$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c} E$

$\vec{\nabla} \times \vec{E} = \frac{i\omega}{c}\vec{B} \Longrightarrow$ $\nabla \times E = \hat{e}_x(\partial E_z/\partial y - \partial E_y/\partial z) + \hat{e}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{e}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$

$$= -\hat{e}_x \frac{\partial E_y}{\partial z} + \hat{e}_z\, \partial E_y/\partial x$$

(4) $i\frac{\omega}{c}\vec{B} = -\hat{e}_x \frac{\partial E_y}{\partial z} + ik\,\hat{e}_z E$

so $\underline{B_y = 0}$ ($\because$ no y variation), using (1) above $\Rightarrow \frac{\partial B_z}{\partial y} = 0$

" (2) " $\Rightarrow \frac{\partial B_x}{\partial y} = 0$

(c) z differential Eqtn

eq(4) $\Rightarrow$ $B_x = -\frac{c}{i\omega}\frac{\partial E_y}{\partial z}$, $B_z = \frac{ck}{\omega}E$, $B_y = 0$

(3) $\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c}E$, now substitute from above

$$-\frac{c}{i\omega}\frac{\partial^2 E_y}{\partial z^2} - \frac{ck}{\omega}\frac{\partial E}{\partial x} = -\frac{i\omega}{c}E \qquad (E = E_0\hat{e}_y e^{i(kx-\omega t)})$$

$$+\frac{ic}{\omega}\frac{\partial^2 E}{\partial z^2} - \frac{ck}{\omega}ikE = -\frac{i\omega}{c}E$$

$$\frac{\partial^2 E}{\partial z^2} - k^2 E = -\frac{\omega^2}{c^2}E$$

$$\frac{\partial^2 E}{\partial z^2} = \left(k^2 - \frac{\omega^2}{c^2}\right)E = -\left(\frac{\omega^2}{c^2} - k^2\right)E = -K^2 E$$

$E_y = 0$ at $z = 0, d$

(d) $E(z) = A\sin Kz + B\cos Kz$ $\qquad (K^2 = \omega^2/c^2 - k^2)$

$E(0) = 0 \Rightarrow B = 0$

$E(d) = 0 \Rightarrow Kd = n\pi \qquad (n = 1, 2 \ldots)$

$$K = \frac{n\pi}{d} = \sqrt{\omega^2/c^2 - k^2}$$

$$\vec{E} = E_0 \sin\left(\frac{n\pi z}{d}\right)e^{i(kx-\omega t)}\,\hat{e}_y$$

$$B_y = 0, \quad B_z = \frac{ck}{\omega}E = \frac{ck}{\omega}E_0\sin\left(\frac{n\pi z}{d}\right)e^{i(kx-\omega t)}$$

$$B_x = -\frac{c}{i\omega}\frac{\partial E_y}{\partial z} = -\frac{c}{i\omega}E_0\frac{n\pi}{d}\cos\left(\frac{n\pi z}{d}\right)e^{i(kx-\omega t)}$$

(e)

Spherical Harmonic
$\nabla^2\phi$ = in spherical coordinates

3

Recursion eqn for P_n

A dielectric sphere of radius R has a permanent polarization $\vec{P}$, which is uniform in direction and magnitude. The polarized sphere gives rise to an electric field. Determine this field ~~both~~ both inside and outside the sphere.

Solution: since $\nabla\cdot\vec{P}=0$ there are no bound charges except at the surface of the sphere. Hence $\nabla^2 U = 0$ everywhere except at the surface. The simplest relevant solutions to Laplace's equation are

Give general expansion?

$$U^{outside} = A r^{-2}\cos\theta \qquad \text{(Dipole potential)}$$

$$U^{inside} = B r \cos\theta \qquad \text{(Constant } \vec{E}\text{)}$$

The constants A, B are found from the boundary conditions at the surface of the sphere

a.) Tangential components of $\vec{E}$ equal:

$$E_{tang} = E_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}U$$

$$E_\theta^{out} = AR^{-3}\sin\theta = B\sin\theta = E_\theta^{in}$$

$$\text{so } A = BR^3$$

b.) The normal comp. of $\vec{D}$ are equal:

$$D_{norm}^{in} = D_r^{in} = \epsilon_0 E_r + P_r = -\epsilon_0\frac{\partial U}{\partial r} + P\cos\theta$$

$$D_{norm}^{out} = 2A\epsilon_0 R^{-3}\cos\theta = -B\epsilon_0\cos\theta + P\cos\theta = D_{norm}^{in}$$

$$3\epsilon_0 B = P \qquad B = \frac{P}{3\epsilon_0} \qquad A = \frac{PR^3}{3\epsilon_0}$$

The fields then are

$$E_\theta^{out} = \frac{PR^3}{3\epsilon_0 r^3}\sin\theta \qquad E_r^{out} = \frac{2PR^3}{3\epsilon_0 r^3}\cos\theta$$

$$E_\theta^{in} = \frac{P}{3\epsilon_0}\sin\theta \qquad E_r^{in} = -\frac{P}{3\epsilon_0}\cos\theta$$

note that $\vec{E}^{in} = -\frac{1}{3\epsilon_0}\vec{P}$, the "depolarizing" field.

QM - Undergrad. $\boxed{\text{Formula for 2nd order PT?} \checkmark}$ 32-8

#4

A quantum mechanical system that has three energy eigenstates is represented by the hamiltonian matrix

$$H = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}$$

where a and b are "small" in the sense that $|a|, |b| \ll E_1, E_2$.

1. Use ordinary stationary - state perturbation theory to find the energy eigenvalues to second order.

2. Find the exact eigenvalues. ~~by diagonalizing the matrix~~.

3. Compare the results from 1.) and 2.) by expanding the exact solution in a power series and keeping the second order terms. Do the two solutions agree? If not, explain carefully what the difficulty is. How would one overcome this problem in a practical calculation?

QM - Solution

1.) Let $H \equiv H_0 + H'$ where $H' = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a^* & b^* & 0 \end{pmatrix}$ is the perturbation. In second order

$$W_m = E_m + H'_{mm} + \sum_{n \neq m} \frac{|H'_{mn}|^2}{E_m - E_n}$$

where $H \psi_m = W_m \psi_m$ and ~~E_0~~ $H_0 u_m = E_m u_m$

$$W_1 = E_1 + \frac{|a|^2}{E_1 - E_2} \qquad W_2 = E_1 + \frac{|b|^2}{E_1 - E_2}$$

$$W_3 = E_2 + \frac{|a|^2 + |b|^2}{E_2 - E_1}$$

2.) The exact eigenvalues are solutions of the equation

$$\det \begin{vmatrix} E_1 - W & 0 & a \\ 0 & E_1 - W & b \\ a^* & b^* & E_2 - W \end{vmatrix} = (E_1 - W)\left[(E_1-W)(E_2-W) - |b|^2 - |a|^2\right] =$$

$$W_1 = E_1 \qquad W_{2,3} = \frac{E_1 + E_2 \pm \sqrt{(E_1-E_2)^2 + 4(|a|^2+|b|^2)}}{2}$$

3.) To second order: $W_2 = E_1 + \dfrac{|a|^2 + |b|^2}{E_1 - E_2}$

$$W_3 = E_2 + \frac{|a|^2 + |b|^2}{E_2 - E_1}$$

The two solutions fail to agree because two of the unperturbed eigenfunctions are degenerate. It is not known a priori what linear combination of these eigenfunctions the exact solution collapses into when the perturbation is removed, hence the normal expansion fails. In a practical calculation one would have to diagonalize a submatrix sufficiently large to contain enough of the perturbation to

7.5

a) The Flux per unit area is

$$J = \text{const} \int_0^\infty dv_x \int_{-\infty}^\infty dv_y \int_{-\infty}^\infty dv_z \, v_x e^{-\frac{1}{2}\frac{m}{kT}(v_x^2+v_y^2+v_z^2)}$$

The molecular density is

$$\frac{N}{V} = \text{const} \int_{-\infty}^\infty dv_x \int_{-\infty}^\infty dv_y \int_{-\infty}^\infty dv_z \, e^{-\frac{1}{2}\frac{m}{kT}(v_x^2+v_y^2+v_z^2)}$$

$$\frac{J}{N/V} = \frac{\int_0^\infty v_x e^{-\frac{1}{2}mv_x^2/kT}}{\int_{-\infty}^\infty e^{-\frac{1}{2}mv_x^2/kT}} = \frac{\sqrt{\frac{2kT}{m}} \int_0^\infty y e^{-y^2} dy}{\int_{-\infty}^\infty e^{-y^2} dy}$$

$$= \frac{\frac{1}{2}\sqrt{\frac{2kT}{m}}}{\sqrt{\pi}}$$

By the ideal gas law $\frac{N}{V} = \frac{P}{kT}$

The total particle flow is $I = J\pi R^2$

$$I = \pi R^2 J = \pi R^2 \frac{P}{kT}\sqrt{\frac{kT}{2\pi m}} = \frac{\pi R^2 P}{\sqrt{2\pi m kT}}$$

$$= cT^t p^p$$

$$c = \frac{\pi R^2}{\sqrt{2\pi m k}} \quad t = -\frac{1}{2} \quad p = 1$$

(b) At a steady state

$$J_{a\to b} = c\,T_a^{-\frac{1}{2}} P_a$$

$$J_{b\to a} = c\,T_b^{-\frac{1}{2}} P_b$$

$$J_{a\to b} = J_{b\to a}$$

$$T_a^{-\frac{1}{2}} P_a = T_b^{-\frac{1}{2}} P_b$$

$$\frac{P_a}{P_b} = \left(\frac{T_a}{T_b}\right)^{\frac{1}{2}}$$

(c) If a spherical atom has a radius a

$$L\,\pi(2a)^2 = \frac{V}{N} = \frac{kT}{P}$$

$$L = \frac{kT\,.\,\cancel{\sigma}}{4P\pi a^2}$$

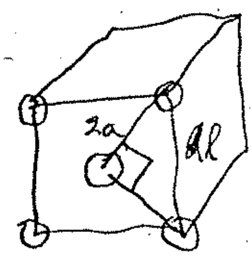
M is atomic weight

N_{av} is Avogadro's number

ℓ is size of unit cubic cell

For a FCC crystal

$$d = \frac{m}{V} = \frac{\frac{4M}{N_{av}}}{\ell^3}$$



ℓ is related to a by

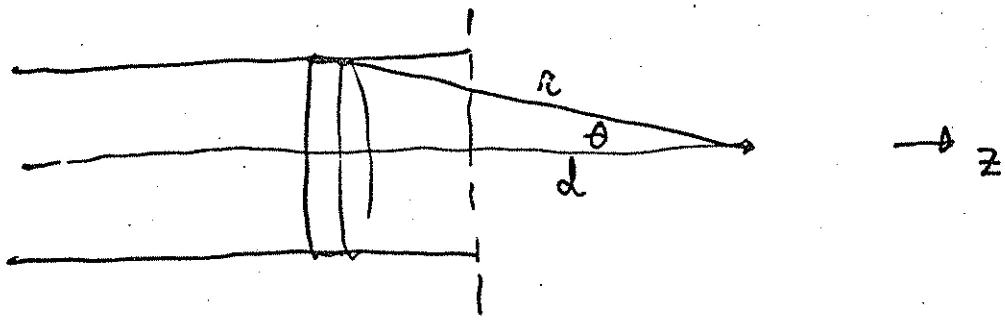
$$2a = \frac{\sqrt{2}\,\ell}{\sqrt{2}} \qquad \ell = 2^{\frac{3}{2}} a$$

$$d = \frac{4M}{N_{av}\cdot 8a^3}$$

$$a^2 = \cancel{4}\left(\frac{M}{2N_{av}d}\right)^{2/3}$$

$$L = \frac{kT}{4\pi P}\left(\frac{M}{2N_{av}d}\right)^{2/3}$$

6a



$$d\vec{B} = \frac{\mu_0 \vec{J} \times \vec{r}}{4\pi r^3}\, dv$$

The contribution of a length dx of the solonoid is (in cylindrical coordin

$$d\vec{B} = \mu_0 \int_0^{2\pi} R\,d\theta \left[\frac{IN\,dz}{L}\right] \frac{\vec{a}_\theta \times \vec{r}}{4\pi r^3}$$

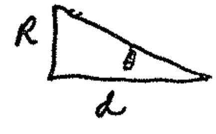
$$\int_0^{2\pi} \vec{a}_\theta \times \vec{r}\; d\theta = 2\pi R\ \vec{a}_z$$ since the contribution from the component of $\vec{r} \parallel z$ cancels out

$$dB = \frac{\mu_0}{2}\frac{R^2 IN}{L}\ \frac{dz}{r^3}$$

$$R = r\sin\theta \qquad\qquad R = -z\tan\theta$$

$$r = \frac{R}{\sin\theta} \qquad\qquad z = \frac{-R}{\tan\theta}$$

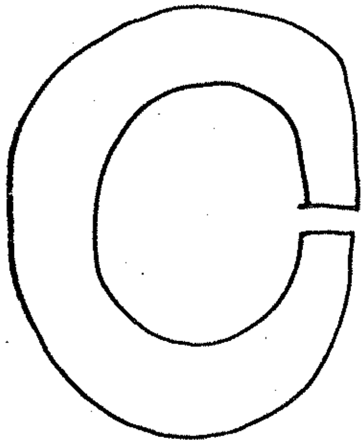
$$dz = \frac{R}{\sin^2\theta}\, d\theta$$



$$B = \frac{\mu_0 R^2 IN}{2L} \int_{\tan^{-1}\frac{R}{d+L}}^{\tan^{-1}\frac{R}{d}} \frac{R\,d\theta}{\sin^2\theta} \times \frac{\sin^3\theta}{R^3}$$

$$= \frac{\mu_0 IN}{2L} \int_{\tan^{-1}\frac{R}{d+L}}^{\tan^{-1}\frac{R}{d}} \sin\theta\, d\theta = \frac{\mu_0 IN}{2L}\left[\frac{d+L}{\sqrt{(d+L)^2+R^2}} - \frac{d}{\sqrt{d^2+R^2}}\right]$$

6b



By Ampere law

$$\oint H\,dl = NI$$

$$= H_a w + H_t(L-w)$$

$$H_a = \frac{B_a}{\mu_0} \qquad H_t = \frac{B_t}{\mu}$$

$B_a = B_t$ since $\vec{n}\cdot\vec{B}$ is ~~continuous across~~ constant across a surface of discontinuity

$$NI = B\left(\frac{w}{\mu_0} + \frac{L-w}{\mu}\right)$$

$$B = \frac{NI}{\frac{L-w}{\mu} + \frac{w}{\mu_0}} = \frac{\mu NI/L}{1 + \frac{w}{L}\left(\frac{\mu}{\mu_0} - 1\right)}$$