PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #31

October 7, 1978

General Instructions

This Comprehensive Examination for Fall 1978 (#31) consists of six problems of equal weight (20 points each). Half of the material is judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show as clearly as possible all your steps so that partial credit can be given liberally in case you do not complete a problem or make errors. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

Possibly useful formulae:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For a harmonic oscillator:

$$\langle n|x|n'\rangle = \sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{n+1}{2}}\,\delta_{n+1,n'} + \sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{n}{2}}\,\delta_{n-1,n'}$$

$$\langle n|p|n'\rangle = -i\sqrt{m\hbar\omega}\sqrt{\frac{n+1}{2}}\,\delta_{n+1,n'} + i\sqrt{m\hbar\omega}\sqrt{\frac{n}{2}}\,\delta_{n-1,n'}$$

$$E_n' = E_n + \langle n|V|n\rangle + \sum_{m\neq n} \frac{|\langle n|V|m\rangle|^2}{E_n - E_m}$$

(10) 1. (a) A diatomic molecule in its electronic ground state has a motion which can frequently be approximated as the sum of three Hamiltonians

$$H_o = H_t + H_r + H_v$$

where the partial Hamiltonians describe the motion of a free particle of mass M, a rotor with a scalar moment of inertia I, and a harmonic oscillator of frequency ω and reduced mass μ. Classically,

$$H_t = P^2/2M$$

$$H_r = P_\theta{}^2/2I$$

$$H_v = p_\delta{}^2/2\mu + \frac{1}{2}\mu\omega^2\delta^2$$

Describe the quantum numbers for the three motions. (Assume the molecule is in a cubic box of length L.) Write an expression for the eigenfunction Ψ^o in terms of the eigenfunctions ψ_t, ψ_r, and ψ_v of the partial Hamiltonians, and express the energy of the eigenstate in terms of the quantum numbers.

([illegible]) (b) A better approximation for the energy of the diatomic molecule is given by the Hamiltonian whose classical expression is

$$H = \frac{p_1{}^2}{2m_1} + \frac{p_2{}^2}{2m_2} + \frac{1}{2}\,k[\,|\vec{r}_1 - \vec{r}_2| - r_o]^2$$

where m_1, $\dot{\vec{r}}_1 = \vec{p}_1/m_1$ describe the mass and velocity of one atom and m_2, $\dot{\vec{r}}_2 = \vec{p}_2/m_2$ describe the other. Derive an expression for the perturbation term H_1 in $H = H_o + H_1$. What is the form of the eigenfunction Ψ in terms of the eigenstates Ψ^o of H^o of part (a). How are I, M, and ω related to m_1, m_2, and k?

2. An astronomer has discovered an interplanetary object and makes measurements which yield the velocity $\vec{v}_0$ and the distance $\vec{r}_0$ with respect to the sun at the time of the discovery. Assume that the motion of the object is affected only by the gravitational force of the sun. Answer the following questions in terms of the given parameters as well as the mass M of the sun and the gravitational constant G.

(10) (a) What are the constants of motion of the object?

(b) What relations determine whether the trajectory is an ellipse, hyperbola, or a parabola?

(10) (c) For each type of possible trajectory, derive the distance of closest approach r_1.

(5) 3. (a) Eigenstates of the Hamiltonian are often called "stationary states". Justify this by showing that if $|\Psi\rangle$ is an eigenstate of energy, then $\langle\Psi|\hat{O}|\Psi\rangle$ is independent of time for <u>any</u> operator which is not explicitly a function of time (i.e., does not contain time in its definition).

(15) (b) In the energy representation of the one-dimensional harmonic oscillator an operator $\hat{A}$ can be defined by

$$\hat{A}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle$$

(i) Is $\hat{A}$ Hermitian?

(ii) Are all the eigenvalues of $\hat{A}$ real?

(iii) Set up the eigenvalue problem for $\hat{A}$ in the above representation.

(iv) Solve this equation for the unnormalized eigenfunctions of $\hat{A}$ in the special case of eigenvalue zero.

4. A one-dimensional oscillator of frequency ω is perturbed by the addition of a potential $v(x) = \frac{1}{2} m\lambda x^2$.

(15) (a) Use non-degenerate perturbation theory to find the energy of the ground state up to <u>second order</u> in λ.

(5) (b) The exact solution is easy to obtain for this potential. Expand the exact ground state energy in powers of λ and compare with the perturbation result of part (a).

(15) 5. (a) Find the electrostatic Green's functions for the infinite half-space defined by $x \geq 0$ and subject to

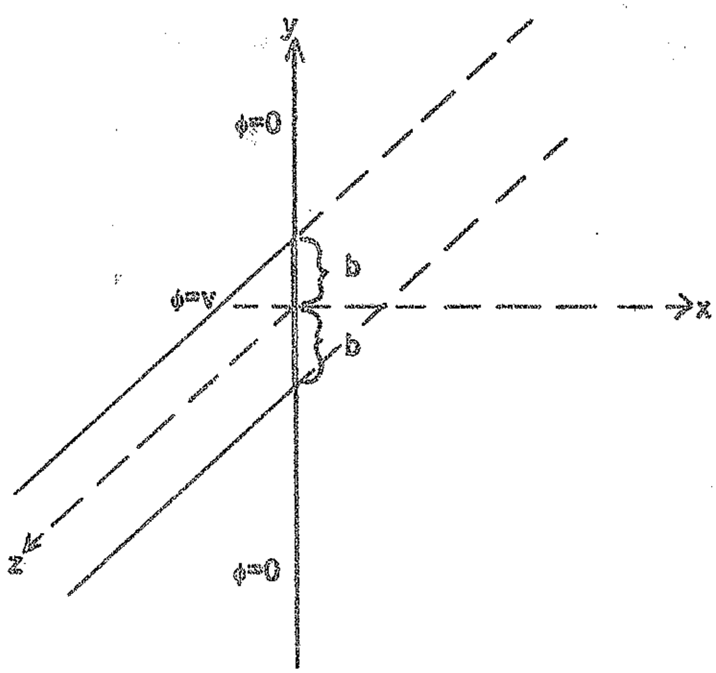
(1) Dirichlet ("first kind") boundary conditions $G^{I}(\vec{x},\vec{x}') = 0$ for $\vec{x}'$ on S.

(2) Neumann ("second kind") boundary conditions $\frac{\partial G^{II}}{\partial n'}(\vec{x},\vec{x}') = 0$ for $\vec{x}'$ on S.

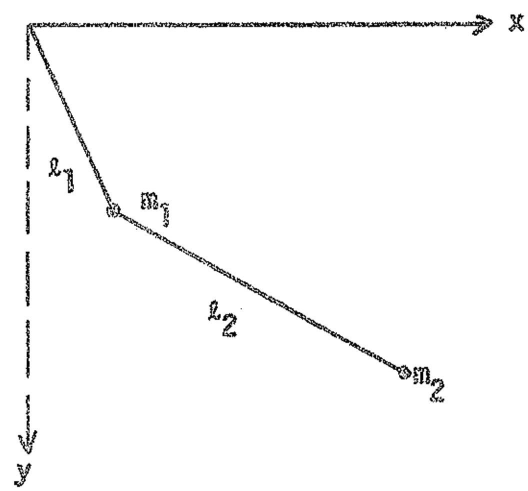
Verify that your functions satisfy the boundary conditions and the defining equation

$$\nabla'^{2} G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

(5) (b) Use the appropriate Green's function from above to solve the following problem: The infinite y-z plane is covered by a conducting sheet. A strip of this sheet of width 2b is raised to a potential V, and is insulated from the rest of the sheet, which is grounded. Find the potential distribution for x>0. NOTE: You need not work out the final integrals.



6. Consider the coplanar double pendulum in a uniform gravitational field



where the two masses are connected by massless rigid rods that are free to move only in the x-y plane.

(7) (a) Define a complete set of generalized coordinates for this device and construct the Lagrangian in terms of these variables.

(6) (b) Find the equations of motion.

(7) (c) For $m_1 = m_2 = m$, $\ell_1 = \ell_2 = \ell$, and small displacements from equilibrium, what are the frequencies of the normal modes?

1 a

$H_t \psi_t = E_t \psi_t$: $k_x L = 2\pi n_x$ $\quad k_x = \frac{2\pi}{L} n_x$ $\quad n_x = \pm 1, \pm 2 \cdots$

$E_t = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$ $\quad k_y L = 2\pi n_y$ $\quad k_y = \frac{2\pi}{L} n_y$ $\quad n_y = \pm 1, \pm 2, \cdots$

$k_z L = 2\pi n_z$ $\quad k_z = \frac{2\pi}{L} n_z$ $\quad n_z = \pm 1, \pm 2 \cdots$

ψ_t has quantum numbers n_x, n_y, n_z, or k_x, k_y, k_z

$H_r \psi_r = E_r \psi_r$ $\quad l = 0, 1, 2 \cdots$

$E_r = \frac{\hbar^2}{2I} l(l+1)$ $\quad m = 0, \pm 1, \pm 2 \cdots \pm l$

ψ_r has quantum numbers l, m

$H_v \psi_v = E_v \psi_v$

$E_v = \hbar\omega(n + \frac{1}{2})$ $\quad n = 0, 1, 2 \cdots$

ψ_v has quantum number n

$$\Psi^0 (n_x, n_y, n_z, l, m, n) = \psi_t(n_x, n_y, n_z)\, \psi_r(l, m)\, \psi_v(n)$$

$$E^0 (n_x, n_y, n_z, l, m, n) = E_t(n_x, n_y, n_z) + E_r(l) + E_v(n)$$

b) If R is the position of the center of mass

$$M\vec{R} = m_1 \vec{r}_1 + m_2 \vec{r}_2 \quad \text{where } M = m_1 + m_2$$

Let $\vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{r}_1 = \vec{r} + \vec{r}_2$

$$M\vec{R} = m_1 \vec{r} + m_1 \vec{r}_2 + m_2 \vec{r}_2 = m_1 \vec{r} + M \vec{r}_2$$

$$\therefore \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

$$\vec{r}_1 = \vec{r} + \vec{r}_2 = \vec{R} + \frac{m_2}{M}\vec{r}$$

Then

$$T = \frac{1}{2}\frac{p_1^2}{m_1} + \frac{1}{2}\frac{p_2^2}{m_2} = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

$$= \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{M}\dot{\vec{r}}^2$$

Therefore

$$H_\pi = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 + \frac{k}{2}(r - r_0)^2$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

If $\vec{r} = r\vec{a}_r$

$$\dot{\vec{r}} = \dot{r}\vec{a}_r + r\frac{d\theta}{dt}\vec{a}_\theta$$

$$\dot{\vec{r}}^2 = \dot{r}^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

$$H = \frac{1}{2}\frac{P^2}{M} + \left[\frac{\mu}{2}\dot{r}^2 + \frac{k}{2}(r - r_0)^2\right] + \frac{\mu}{2}r^2\left(\frac{d\theta}{dt}\right)^2$$

where $\vec{P} = M\dot{\vec{R}}$

Let $\xi = r - r_0$ $\quad r^2 = (r_0 + \xi)^2 = r_0^2 + 2\xi r_0 + \xi^2$

$$p_\xi = \mu\dot{\xi} = \mu\dot{r}$$

$$I = \mu r_0^2$$

$$p_\theta = I\frac{d\theta}{dt}$$

$$H = \frac{P^2}{2M} + \left(\frac{p_\xi^2}{2\mu} + \frac{k}{2}\xi^2\right) + \frac{p_\theta^2}{2I}$$

$$+ \frac{\mu}{2}\left(\frac{d\theta}{dt}\right)^2\left[2r_0\xi + \xi^2\right]$$

$$= H_t + H_r + H_n + H_1 = H_0 + H_1 \qquad \text{where } k = \mu\omega^2$$

where $H_1 = \dfrac{\mu}{2}\left(\dfrac{d\theta}{dt}\right)^2\left(2r_0\xi + \xi^2\right) = \dfrac{p_\theta^2}{2I}\left(2\dfrac{\xi}{r_0} + \xi^2\right)$

Ψ will be of the form

$$\Psi = \sum_{n,l,m} c_{nlm}\,\Psi^0(n, l, m, n_x, n_y, n_z)$$

since H_1 contains coordinates of H_t and H_r

CX #31 Solution

#2

a) Constants of motion

$$e = \frac{E}{m} = \frac{1}{2}v_0^2 - \frac{GM}{R_0}$$ where m is mass of object

$$\vec{\ell} = \frac{\vec{L}}{m} = \vec{v}_0 \times \vec{R}_0$$

b) If $e > 0$, then

$\frac{1}{2}v_0^2 > \frac{GM}{r_0}$

$$e = \frac{1}{2}v^2 - \frac{GM}{R} > 0 \text{ when } r \to \infty$$
$$\cong \frac{1}{2}v^2$$

Since the force $-\frac{GM}{r^2} \to 0$, the trajectory approaches a straight line asymptotically, ~~and~~ which corresponds to a hyperbolic ~~orbit~~ trajectory

$\frac{1}{2}v_0^2 < \frac{GM}{r_0}$ If $e < 0$, r cannot ever be infinite, so the trajectory is elliptical

$\frac{1}{2}v_0^2 = \frac{GM}{r_0}$ If $e = 0$, $v \to 0$ when $r \to \infty$, so that the trajectory is neither elliptical nor hyperbolic. It is parabolic

c) At the point of closest approach, the ~~accelerations~~ acceleration (change in $\vec{v}$) must be purely radial, so that

$$t = \frac{1}{2}v^2 = \frac{1}{2}\left(R_1\frac{d\theta}{dt}\right)^2$$

But at that point $\ell = R_1\left(r_1\frac{d\theta}{dt}\right)$

$$\text{Therefore } \left(r_1\frac{d\theta}{dt}\right)^2 = \left(\frac{\ell}{R_1}\right)^2$$

$$e = \frac{1}{2}\left(\frac{\ell}{r_1}\right)^2 - \frac{GM}{R_1}$$

Writing $x = 1/r_1$

$$\frac{\ell^2}{2}x^2 - MGx - e = 0$$

$$x = \frac{MG}{\ell^2} \pm \frac{\sqrt{M^2g^2+2\ell^2 e}}{\ell^2} = \frac{Mg}{\ell^2}\left[1 \pm \sqrt{1+\frac{2\ell^2 e}{M^2G^2}}\right]$$

For a parabola $e=0$

$$x = \frac{1}{r_1} = \frac{2Mg}{\ell^2}$$

(The other solution is the turning point at $r=\infty$)

$$r_1 = \frac{\ell^2}{2Mg}$$

For a hyperbola, $e>0$

$$x = \frac{1}{r_1} = \frac{Mg}{\ell^2}\left[1 + \sqrt{1+2\frac{\ell^2 e}{M^2G^2}}\right]$$

(The $-$ sign solution yields $r_1<0$)

$$r_1 = \frac{\ell^2/Mg}{1+\sqrt{1+2\frac{\ell^2 e}{M^2G^2}}}$$

For an ellipse $e<0$

$$x = \frac{Mg}{\ell^2}\left[1 + \sqrt{1+2\frac{\ell^2 e}{M^2G^2}}\right]$$

$$r_1 = \frac{\ell^2/Mg}{1+\sqrt{1+2\frac{\ell^2 e}{M^2G^2}}}$$

(The minus-sign solution yields a larger value of r, which corresponds to the farthest point of the orbit

① Solution to Problem 3

ⓐ Show $\langle\psi|\hat{O}|\psi\rangle$ independent of time

In coordinate space representation; $\langle\psi|\hat{O}|\psi\rangle = \int \psi^*(\underset{\sim}{r},t)\, O(\underset{\sim}{r},t)\, \psi(r,t)\, d^3r$

For Stationary state $\psi(\underset{\sim}{r},t)$;

$$H\psi = i\hbar\, \partial\psi/\partial t = E\psi$$

$$\psi(\underset{\sim}{r},t) = e^{-iEt/\hbar}\,\phi(\underset{\sim}{r})$$

so

$$\langle\psi|\hat{O}|\psi\rangle = \int e^{iEt/\hbar} R^*(r)\,\hat{O}(\underset{\sim}{r},t)\, R(r)\, e^{-iEt/\hbar}\, d^3r$$

$$= \int |R(r)|^2\, \hat{O}(\underset{\sim}{r},t)\, d^3r \quad .$$

So; if $\hat{O}$ does not depend explicitly on time, $\langle\psi|\hat{O}|\psi\rangle$ is stationary, i.e. independent of time.

ⓑ $$\hat{A}|n\rangle = \sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle$$

ⓘ $\hat{A}$ is Hermitian if $\hat{A} = \hat{A}^*$ (Hermitian adjunt)

i.e. $\langle m|A|n\rangle = \langle n|A|m\rangle^* = \langle An|m\rangle$

$$\langle m|\hat{A}|n\rangle = \langle m|N+1\rangle\sqrt{n+1} - \sqrt{N}\,\langle m|n-1\rangle = \sqrt{n+1}\,\delta_{m,n+1} - \sqrt{N}\,\delta_{m,n-1} \qquad (\alpha)$$

$$\langle Am|n\rangle = \sqrt{m+1}\,\langle m+1|n\rangle - \sqrt{m}\,\langle m-1|n\rangle \begin{cases} = \sqrt{m+1}\,\delta_{m+1,n} - \sqrt{m}\,\delta_{m-1,n} \\ = \sqrt{n}\,\delta_{m,n+1} - \sqrt{n+1}\,\delta_{m,n-1} \qquad (\beta)\end{cases}$$

where we employ orthonormality of harmonic oscillator wavefunctions

Since eq. $(\alpha) \neq (\beta)$ $\hat{A}$ is not hermitian $\left[\begin{matrix}\text{or } A^{\dagger} = a^{\dagger} - a \\ \text{so} \\ A^{\dagger} = -A \neq A\end{matrix}\right]$

ⓘⓘ NO, since $\hat{A}$ is not Hermitian all its eigenvalues are not real

ⓘⓘⓘ
$$\hat{A}\psi = A_0\psi$$
$$\psi = \sum_{n=0}^{\infty} a_n|n\rangle$$

$$\hat{A}\psi = \sum_n a_n \hat{A}|n\rangle = \sum_n a_n\left\{\sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle\right\} = A_0\psi = A_0\sum_n a_n|n\rangle$$

(ii) #3 contd

(iv) If $A_0 = 0$

$$\sum_{n=0}^{\infty} a_n \left\{ \sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle \right\} = 0$$

$$\sum_{m=1} a_{m-1} \sqrt{m}\,|m\rangle - \sum_{m=0} a_{m+1} \sqrt{m+1}\,|m\rangle = 0$$

$$\boxed{a_{m+1} = \sqrt{\frac{m}{m+1}}\, a_{m-1}}$$

$\therefore a_0$ arbitrary, $a_1 = 0$

Solution Q.M. problem 4

ⓐ $H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$; unperturbed Hamiltonian

$V = \frac{1}{2} m \lambda x^2$; perturbation

H_0 has eigenvalues $E_n = (n+1/2)\hbar\omega$.

energy with perturbation: $E_n' = E_n + \langle n|V|n\rangle + \sum_{m\neq n} \frac{|\langle n|V|m\rangle|^2}{E_n - E_m}$

for $n=0$, ground state,

$$E_0' = \frac{\hbar\omega}{2} + \frac{1}{2} m\lambda \langle 0|x^2|0\rangle + \frac{m^2\lambda^2}{4} \sum_{m\neq 0} \frac{|\langle 0|x^2|m\rangle|^2}{\frac{\hbar\omega}{2} - \hbar\omega(m+1/2)}$$

We could use virial theorem for $\langle 0|x^2|0\rangle$, yet since we need $\langle 0|x^2|m\rangle$ too, will use matrix algebra.

Since $|0\rangle$ has even parity, so does x^2, only m even contribute.

$$\langle 0|x^2|m\rangle \equiv \langle \hat{x}0|\hat{x}m\rangle = \sum_{n'} \langle \hat{x}0|n'\rangle\langle n'|\hat{x}|m\rangle = \sum_{n'} \langle 0|\hat{x}|n'\rangle\langle n'|\hat{x}|m\rangle$$

So we can form matrix $\langle n|\hat{x}|m\rangle$ and multiply, and keep elements

$$\langle n|x|m\rangle = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & \end{pmatrix}$$

$$\langle n|x|m\rangle\langle n|x|m\rangle = \frac{\hbar}{m\omega}\frac{1}{2} \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 3 & 0 & \sqrt{6} & \end{pmatrix}$$

read off $\langle 0|x^2|0\rangle$
$\langle 0|x^2|2\rangle$

$$E = \frac{\hbar\omega}{2}\left\{1 + \frac{\lambda}{2\omega^2} - \frac{1}{8}\frac{\lambda^2}{\omega^4} + \cdots\right\}$$

ⓑ $E_{xact} = \frac{1}{2}\hbar\sqrt{\omega^2+\lambda} = \frac{1}{2}\hbar\omega\left[1 + \frac{1}{2}\frac{\lambda}{\omega^2} - \frac{1}{8}\frac{\lambda^2}{\omega^4} \cdots\right\}$

Problem (5) Solution :

a.) The Green's function $G(\bar{x}, \bar{x}')$ can be thought of as a superposition of $1/r$ potentials due to a point charge at $\bar{x}'$ and various "image charges" chosen to make G satisfy the boundary conditions.

To make $G^{I}(\bar{x}, \bar{x}') = 0$ for $x' = 0$ let

$$G^{I} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2} - \left[(x+x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}$$

To satisfy Neumann boundary conditions

$$G^{II} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2} + \left[(x+x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}$$

Since $\dfrac{\partial G^{II}}{\partial x'} = (x-x')\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-3/2} - (x+x')\left[(x+x')^2 + (y-y')^2 + (z-z')^2\right]^{-3/2}$

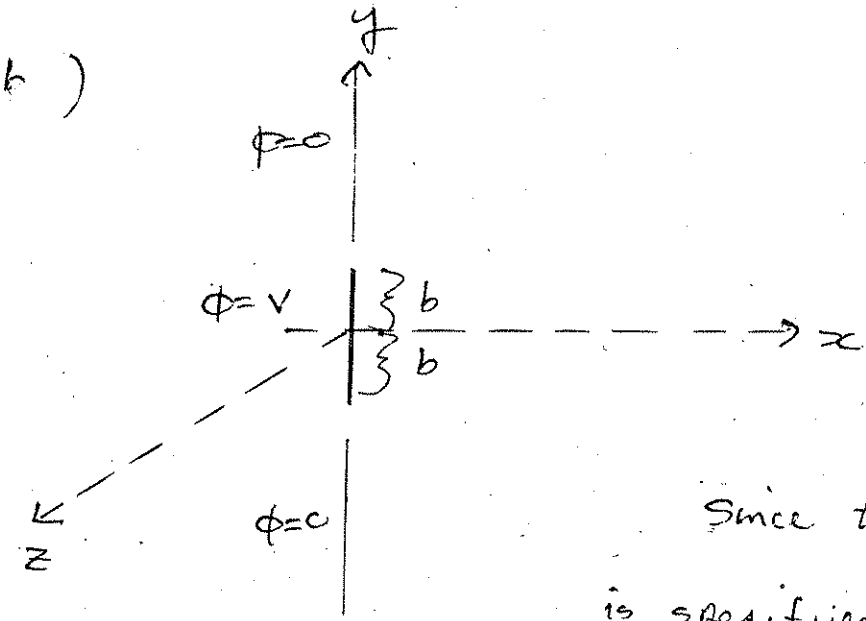
and $\left.\dfrac{\partial G^{II}}{\partial x'}\right|_{x'=0} = 0$

To show what happens at $\bar{x} = \bar{x}'$ translate the origin to x' and use spherical coordinates. Integrate $\nabla^2(1/r)$ over a small volume around $\bar{x}'$

$$\int_V \nabla^2 \tfrac{1}{r}\, dV = \int_V \nabla \cdot \left(\nabla \tfrac{1}{r}\right) dV = \int_S \bar{n} \cdot \nabla\left(\tfrac{1}{r}\right) da$$

$$= \int_S \frac{\partial}{\partial r}\left(\frac{1}{r}\right) r^2 d(\cos\theta)\, d\phi = -4\pi$$

b)



Since the potential on the boundary is specified the problem is of the Dirichlet type:

$$\Phi(x) = -\frac{1}{4\pi} \oint_S \Phi(x') \frac{\partial G}{\partial n'} \, da'$$

$$= -\frac{V}{4\pi} \int_{-\infty}^{+\infty} dz' \int_{-b}^{b} dy' \, 2x \left[x^2 + (y-y')^2 + (z-z')^2 \right]^{-3/2}$$

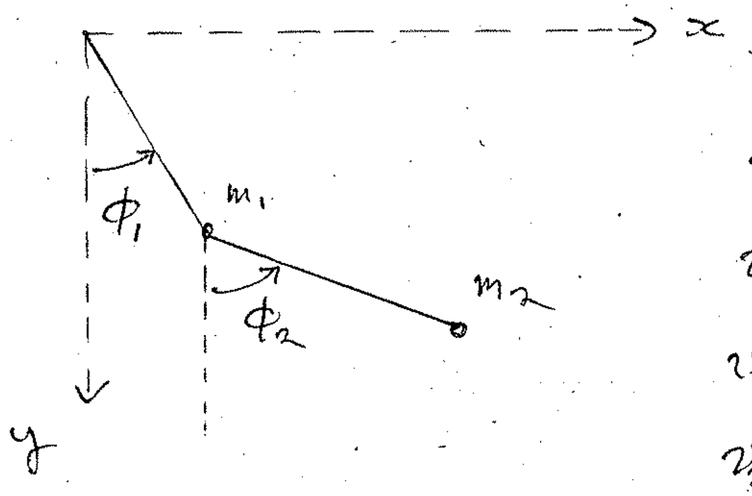
$$= -\frac{Vx}{2\pi} \int_{-\infty}^{+\infty} dz' \left[\frac{y-y'}{[x^2+(z-z')^2][x^2+(y-y')^2+(z-z')^2]^{1/2}} \right]_{-b}^{b}$$

$$= \frac{V}{\pi} \left[\frac{1}{y+b} \tan^{-1}\left(\frac{y+b}{x}\right) - \frac{1}{y-b} \tan^{-1}\left(\frac{y-b}{x}\right) \right]$$

Problem (6)

Solution :) One possible choice of variables is as follows:

a.)



$$v_{1x} = l_1 \dot{\phi}_1 \cos\phi_1$$

$$v_{1y} = -l_1 \dot{\phi}_1 \sin\phi_1$$

$$v_{2x} = v_{1x} + l_2 \dot{\phi}_2 \cos\phi_2$$

$$v_{2y} = v_{1y} + l_2 \dot{\phi}_2 \sin\phi_2$$

$$T = \tfrac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_2 - \phi_1) m_2$$

$$V = -(m_1 + m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

and $L = T - V$

b.)

$$\frac{\partial T}{\partial \phi_1} = l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 m_2 \sin(\phi_2 - \phi_1)$$

$$\frac{\partial T}{\partial \phi_2} = -l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 m_2 \sin(\phi_2 - \phi_1)$$

$$\frac{\partial T}{\partial \dot{\phi}_1} = (m_1 + m_2) l_1^2 \dot{\phi}_1 + l_1 l_2 m_2 \dot{\phi}_2 \cos(\phi_2 - \phi_1)$$

$$\frac{\partial T}{\partial \dot{\phi}_2} = m_2 l_2^2 \dot{\phi}_2 + l_1 l_2 m_2 \dot{\phi}_1 \cos(\phi_2 - \phi_1)$$

$$\frac{\partial V}{\partial \phi_1} = (m_1 + m_2) g l_1 \sin\phi_1$$

$$\frac{\partial V}{\partial \phi_2} = m_2 g l_2 \sin\phi_2$$

The equations of motion then are

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_1}\right) - \frac{\partial L}{\partial \phi_1} = (m_1+m_2)\, l_1^2 \ddot\phi_1 + l_1 l_2 m_2 \ddot\phi_2 \cos(\phi_2-\phi_1)$$
$$- l_1 l_2 m_2 \dot\phi_2^2 \sin(\phi_2-\phi_1) + (m_1+m_2) g l_1 \sin\phi_1$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_2}\right) - \frac{\partial L}{\partial \phi_2} = m_2 l_2^2 \ddot\phi_2 + l_1 l_2 m_2 \ddot\phi_1 \cos(\phi_2-\phi_1)$$
$$+ l_1 l_2 m_2 \dot\phi_1^2 \sin(\phi_2-\phi_1) + m_2 g l_2 \sin\phi_2$$

c.) Let $m_1 = m_2 = m$ $\quad l_1 = l_2 = l$ and keep terms linear in ϕ.

$$2\ddot\phi_1 + \ddot\phi_2 + \frac{2g}{l}\phi_1 = 0$$
$$\ddot\phi_1 + \ddot\phi_2 + \frac{g}{l}\phi_2 = 0$$

These will have a solution of the form $\phi = \phi_0 e^{i\omega t}$ if the equations

$$-2\omega^2\phi_{10} - \omega^2\phi_{20} + \frac{2g}{l}\phi_{10} = 0$$
$$-\omega^2\phi_{20} - \omega^2\phi_{10} + \frac{g}{l}\phi_{20} = 0$$

have algebraic solutions for ϕ_{10} and ϕ_{20}. ie if

$$\begin{vmatrix} \left(\frac{2g}{l} - 2\omega^2\right) & -\omega^2 \\ -\omega^2 & \left(\frac{g}{l} - \omega^2\right) \end{vmatrix} = 0$$

$$\omega^4 - 4\frac{g}{l}\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

or $\omega_\pm^2 = \frac{g}{l}\left(2 \pm \sqrt{2}\right)$