PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #30

April 8, 1978

General Instructions

This Comprehensive Examination for Spring 1978 (#30) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

1. A projectile is to be fired in the x-y plane (y-axis vertical) with muzzle velocity v_0 to hit a target at the point $x = x_0$, $y = 0$.

(5) (a) Neglecting air resistance, find the correct angle of elevation of the gun.

(10) (b) Assuming that air resistance produces a velocity dependent force of the form $-b\, d\vec{r}/dt$ find the equations of motion and solve them.

(5) (c) Assuming that air resistance is very weak, i.e. $bx/mv \ll 1$, find the first-order correction to the angle of elevation.

2. Given two containers of water of equal mass M and specific heat C, one at an absolute temperature T_1 and the other at T_2 ($T_2 < T_1$);

(10) (a) Determine the change in entropy and the final temperature which occurs on mixing the contents of the containers.

(10) (b) Determine the maximum work which can be done, and the final temperature of the water, if a reversible heat engine is operated between these heat reservoirs.

3. Using the method of images, discuss the problem of a point charge q inside a hollow, grounded, conducting sphere of radius a. Find

(10) (a) The potential inside the sphere.

(6) (b) The induced surface charge density.

(4) (c) The total induced charge.

4. Two identical quantum mechanical systems, (1) and (2), when isolated from each other have time-independent eigenstates ψ_1 and ψ_2 which obey the equations

$$H_1\psi_1 = E_0\psi_1 \quad \text{and} \quad H_2\psi_2 = E_0\psi_2 \quad .$$

The systems are placed in contact, with the result that the new Hamiltonian is

$$H = H_1 + H_2 + V \quad ,$$

where V is relatively small. In analyzing the coupled system we make the following assumptions:

(i) ψ_1 and ψ_2 are orthogonal.

(ii) The matrix elements $\langle\psi_1|V|\psi_1\rangle$, $\langle\psi_2|V|\psi_2\rangle$, $\langle\psi_1|H_2|\psi_1\rangle$, and $\langle\psi_2|H_1|\psi_2\rangle$ all vanish.

(iii) $v = \langle\psi_1|V|\psi_2\rangle$ is finite.

(a) Derive the energies and eigenfunctions of the perturbed system, using the variational principle with a wave function of the form

$$\psi = c_1\psi_1 + c_2\psi_2 \quad .$$

(10) (b) If, at time $t = 0$, $|\Psi|^2 = |\psi_1|^2$, determine the behavior of $\Psi(t)$. Hint: If

$$\Psi(t) = \sum_{n=0}^{\infty} c_n(t)\Psi_n^{(0)}(t)$$

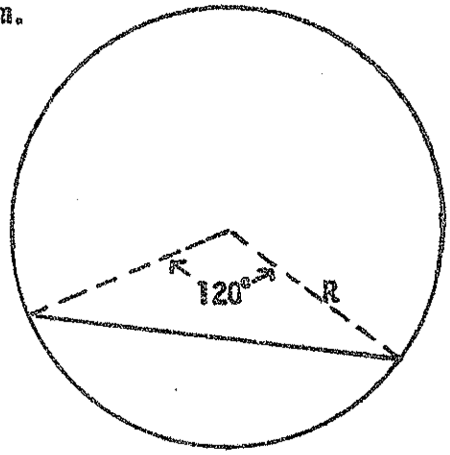
then

$$\frac{\partial c_m}{\partial t} = -\frac{i}{\hbar}\sum_{n=0}^{\infty} c_n(t)\exp\left[\frac{i}{\hbar}(E_m - E_n)t\right] v_{nm} \ ,$$

where $\quad v_{nm} = \langle\psi_n \ |V|\psi_m \ \rangle$

and $\quad \Psi_n^{(0)}(t) = \psi_n \exp\left[-\frac{iE_n t}{\hbar}\right]$.

( ) 5. A uniform rod slides with its ends on a smooth vertical circle. The rod subtends an angle of 120° at the center of the circle, as shown in the figure; and friction is negligible. Determine the Lagrangian and equation(s) of motion for this system; and solve for motion near equilibrium.



(10) 6. (a) Derive the selection rule(s) for electric dipole transitions between vibrational levels in a diatomic molecule. [Hints: (1) The wave functions for a one-dimensional simple harmonic oscillator are given by

$$U_n(x) = N_n \exp[-\alpha^2 x^2/2] H_n(\alpha x) ,$$

where $\alpha^2 = m\omega/\hbar$, N_n is a normalization factor, and $H_n(\alpha x)$ is the nth Hermite polynomial. (2) The recurrence relation for Hermite polynomials is given by

$$H_{n+1}(\xi) - 2\xi H_n(\xi) + 2nH_{n-1}(\xi) = 0 .]$$

(10) (b) In the He^+ ion, the spin and orbital angular momenta of the electron combine to form the total angular momentum, $\vec{J}$, of each fine structure level. In the presence of an applied magnetic field the relative magnitudes of the Zeeman splittings of different fine structure levels is governed by the magnetic moments, $\vec{\mu}_j$. Choosing a small but nonzero value of orbital angular momentum, illustrate in a vector diagram the relation between $\vec{J}$ and $\vec{\mu}_j$. Be sure that your diagram is clearly explained. Also, explain the meaning of the term "Landé g-factor".

Dynamics - UG

1\. a.) At the moment of impact

$$x = x_0 = v_{0_x} t$$

$$z = 0 = v_{0_y} t - \frac{1}{2} g t^2$$

Eliminate t: $v_{0_y} v_{0_x} = \frac{1}{2} g x_0$

$$\sin 2\theta = g x_0 / v_0^2$$

b.) $m\ddot{x} = -b\dot{x}$

$$m\ddot{y} = -mg - b\dot{y}$$

or $\dot{x} = v_{0_x} e^{-bt/m}$

$$x = \frac{m v_{0_x}}{b}\left(1 - e^{-bt/m}\right)$$

$$v_y = \left(\frac{mg}{b} + v_{0_y}\right) e^{-bt/m} - \frac{mg}{b}$$

$$y = \left(\frac{m^2 g}{b^2} + \frac{m v_{0y}}{b}\right)\left(1 - e^{-bt/m}\right) - \frac{mg}{b} t$$

c.) Eliminate t:

$$e^{-bt/m} = 1 - bx / m v_{0_x}$$

$$y = \left(\frac{mg}{b v_{0_x}} + \frac{v_{0y}}{v_{0x}}\right) x + \frac{m^2 g}{b^2} \ln\left(1 - \frac{bx}{m v_{0_x}}\right)$$

Since $\ln(1-\epsilon) = -\epsilon - \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} - \dots$

$$y = \frac{v_{y_0}}{v_{0_x}} x - \frac{g x^2}{2 v_{0_x}^2} - \frac{b g x^3}{3 m v_{0_x}^3} - \dots$$

at the moment of impact

$$v_{0y} \approx \frac{g x_0}{2 v_{0x}} + \frac{b g x_0^2}{3 m v_{0x}^2}$$

$$\sin 2\theta \approx \frac{g x_0}{v_0^2} + \frac{2 b g x_0^2}{3 m v_0^2 \cos\theta_0}$$

Let $\theta = \theta_0 + \delta$ where $\sin 2\theta_0 = g x_0 / v_0^2$

Then to first order in the small parameters

$$\sin 2\theta \approx \sin 2\theta_0 + 2\delta \cos 2\theta_0$$

$$\delta \approx \frac{b g x_0^2}{3 m v_0^2 \cos\theta_0 \cos 2\theta_0}$$

2 a

$$CM(T_f - T_2) = CM\,(T_1 - T_f)$$

$$T_f = \frac{T_1 + T_2}{2}$$

$$\Delta S = \int_{T_f}^{T_1} \frac{đQ_1}{T} + \int_{T_f}^{T_2} \frac{đQ_2}{T}$$

where $đQ$ is the heat added to an external reservoir at equilibrium with a container of M grams of water in the process of restoring each container to its original condition by a reversible process

$$đQ_i = -MC\,dT_i$$

$$\cancel{đQ_2 = -MC\,dT}$$

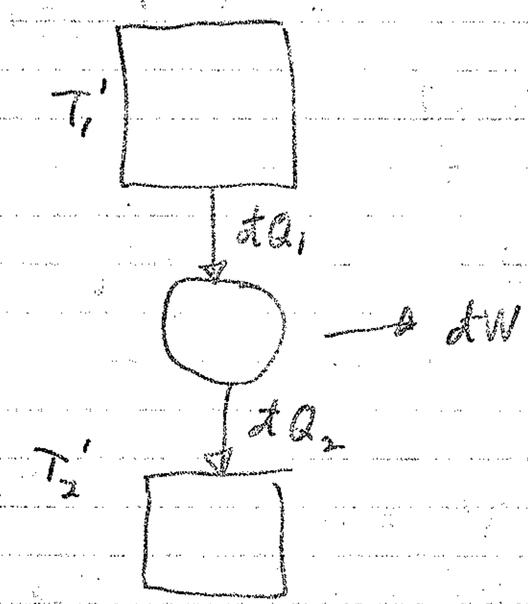
$$\Delta S = -\int_{T_f}^{T_1} \frac{CM\,dT}{T} - \int_{T_f}^{T_2} \frac{CM\,dT}{T}$$

$$= CM\left[\ln\frac{T_f}{T_1} + \ln\frac{T_f}{T_2}\right]$$

$$= CM \ln \frac{T_f^2}{T_1 T_2}$$

$$= CM \ln \frac{(T_1+T_2)^2}{4T_1T_2}$$

b



$$đW = đQ_1 - đQ_2$$

$$\frac{đQ_1}{T_1'} = \frac{đQ_2}{T_2'} \quad \text{for a reversible process}$$

$$đQ_1 = -CM\,dT_1'$$

$$dQ_2 = +CM\,dT_2'$$

$$-\frac{dT_1'}{T_1'} = \frac{dT_2'}{T_2'}$$

$$-\int_{T_1}^{T_f} \frac{dT_1'}{T_1'} = \int_{T_2}^{T_f} \frac{dT_2'}{T_2'}$$

$$-\ln \frac{T_f}{T_1} = \ln \frac{T_f}{T_2}$$

$$T_f = \sqrt{T_1 T_2}$$

$$\Delta Q_1 = \int_{T_1}^{T_f} dQ_1 = -CM \int_{T_1}^{T_f} dT = CM\,(T_1 - T_f)$$

$$\Delta Q_2 = \int_{T_2}^{T_f} dQ_1 = CM\,(T_f - T_2)$$

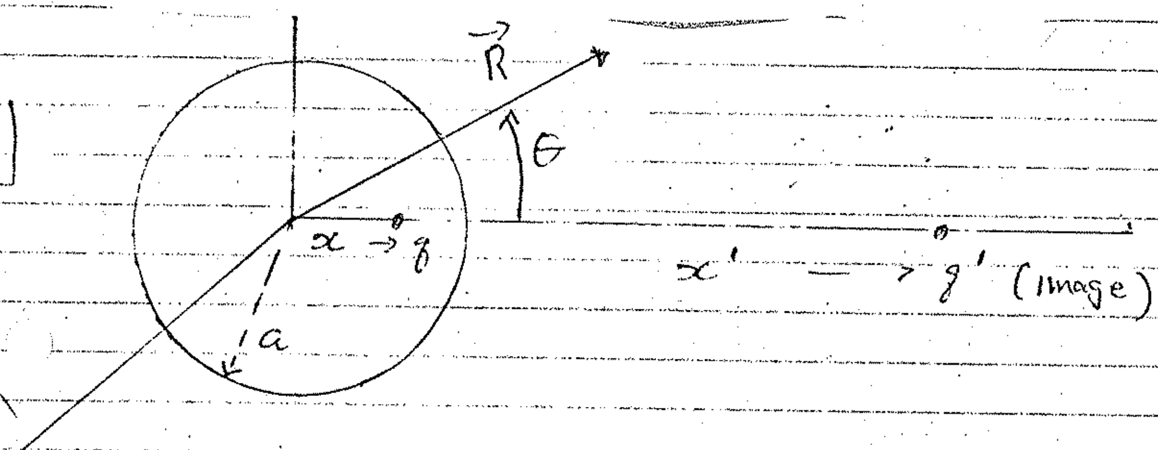
$$\Delta W = \Delta Q_1 - \Delta Q_2 = CM\,(T_1 - T_f - T_f + T_2)$$

$$= CM\left[T_1 + T_2 - 2\sqrt{T_1 T_2}\right]$$

$$= CM\left(\sqrt{T_1} - \sqrt{T_2}\right)^2$$

Grad E + M #3. C-30, Spg, 1978

1]



$$\Phi(R) = \frac{q}{[x^2+R^2-2Rx\cos\theta]^{1/2}} + \frac{q'}{[x'^2+R^2-2Rx'\cos\theta]^{1/2}}$$

we choose x' and q' to force $\Phi(R=a)=0$

$$\Phi(a) = \frac{q}{a\left[\frac{x^2}{a^2}+1-\frac{2x}{a}\cos\theta\right]^{1/2}} + \frac{q'}{x'\left[1+\frac{a^2}{x'^2}-\frac{2a}{x'}\cos\theta\right]^{1/2}}$$

This must vanish for all θ. Hence

$$\frac{x}{a} = \frac{a}{x'} \quad \text{and} \quad \frac{q}{a} = -\frac{q'}{x'}$$

then

$$\Phi(R) = \frac{q}{[x^2+R^2-2Rx\cos\theta]^{1/2}} - \frac{q}{\left[\frac{R^2x^2}{a^2}+a^2-2Rx\cos\theta\right]^{1/2}}$$

The charge density ~~σ~~ is given by the radial derivative

$$\sigma = -\frac{1}{4\pi}\frac{\partial \Phi}{\partial R} = \frac{q}{4\pi a^2}\left(\frac{a}{x}\right)\frac{(1 - a^2/x^2)}{\left[1 + a^2/x^2 - 2a/x \cos\theta\right]^{3/2}}$$

Outside the sphere $\vec{E} = 0$ hence

$$\oint_S \vec{E}\cdot\vec{n}\, da = 4\pi \sum_i q_i = 0$$

for any surface S outside the sphere. So the charge on the sphere $\int \sigma\, da = -q$.

#4

#9

a)

$$\langle E\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$= \frac{\langle c_1\psi_1 + c_2\psi_2 | H_1 + H_2 + V | c_1\psi_1 + c_2\psi_2\rangle}{\langle c_1\psi_1 + c_2\psi_2 | c_1\psi_1 + c_2\psi_2\rangle}$$

$$= \frac{c_1^2 E_0 + c_2^2 E_0 + 2c_1 c_2 v}{c_1^2 + c_2^2}$$

Let $c_2/c_1 = \lambda$

$$\langle E\rangle = \frac{E_0(1+\lambda^2) + 2\lambda v}{1+\lambda^2} = E_0 + \frac{2\lambda}{1+\lambda^2} v$$

$$\frac{\partial\langle E\rangle}{\partial\lambda} = 0 = \frac{2v}{1+\lambda^2} - \frac{4\lambda^2 v}{(1+\lambda^2)^2}$$

$$1+\lambda^2 - 2\lambda^2 = 0$$

$$\lambda = \pm 1$$

$$\psi = \frac{1}{\sqrt{2}}\psi_1 \pm \frac{1}{\sqrt{2}}\psi_2 \qquad E = E_0 \pm v$$

b)

$$\Psi = c_1(t)\Psi_1 + c_2(t)\Psi_2$$

$$\frac{\partial c_1}{\partial t} = -\frac{i}{\hbar} c_2 v \qquad \frac{\partial c_2}{\partial t} = -i c_1 v$$

$$\frac{d^2 c_1}{dt^2} = -\frac{i}{\hbar} v \frac{dc_2}{dt} = -\frac{v^2}{\hbar^2} c_1 \qquad \frac{d^2 c_2}{dt^2} = -\frac{v^2}{\hbar^2} c_2$$

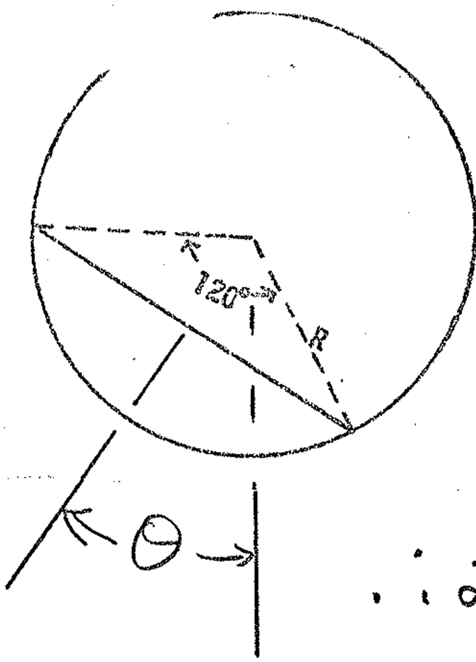
$$c_1 = a_1 \cos\frac{vt}{\hbar} + b_1 \sin\frac{vt}{\hbar} \qquad c_2 = a_2\cos\frac{vt}{\hbar} + b_2 \sin\frac{vt}{\hbar}$$

With boundary conditions $c_1(0) = 1 \quad c_2(0) = 0 \quad c_1^2 + c_2^2 = 1$

$$c_1 = \sin(vt/\hbar) \qquad c_2 = \cos(vt/\hbar$$

$$\Psi = \sin\left(\frac{vt}{\hbar}\right) \psi_1 e^{-iE_0 t/\hbar} + \cos\left(\frac{vt}{\hbar}\right) \psi_2 e^{-iE_0 t/\hbar}$$

#5, C-30, Spg 1978



$T = \frac{1}{2}M\left(\frac{R}{2}\right)^2\dot{\theta}^2 + \frac{1}{2}I_c\dot{\theta}^2,$

where $I_c = 2\rho\int_0^{\frac{\sqrt{3}}{2}R} r^2\,dr = \frac{2\rho}{3}\left(\frac{\sqrt{3}}{2}R\right)^3.$

$M = \sqrt{3}R\rho$; ∴, $I_c = \frac{2}{3}\left(\frac{\sqrt{3}}{2}R^3\right)\frac{M}{\sqrt{3}R},$

or, $I_c = \frac{MR^2}{4}$. $V = \frac{MgR}{2}(1-\cos\theta)$;

$\therefore \mathcal{L} = \frac{MR^2\dot{\theta}^2}{8} + \frac{MR^2\dot{\theta}^2}{8} - \frac{MgR}{2}(1-\cos\theta).$

$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{MR^2\dot{\theta}}{2}$, $\frac{\partial\mathcal{L}}{\partial\theta} = -\frac{MgR}{2}\sin\theta$; $\therefore \ddot{\theta} + \frac{g}{R}\sin\theta = 0.$

For $\theta \ll 1$ $\ddot{\theta} \simeq -\frac{g}{R}\sin\theta$, which yields S.H.M. with angular frequency $\omega = \sqrt{g/R}$.

#6. a) The dipole transition matrix element is given by

$M_{n,n'} \sim \langle n|x|n'\rangle$, or, $M_{n,n'} \sim \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} H_n(\alpha x)\, x\, H_{n'}(\alpha x).$

From the recurrence formula, $\alpha x H_{n'}(\alpha x) = n' H_{n'-1} + H_{n'+1}$;

$\therefore, M_{n,n'} \sim n'\underbrace{\langle n|n'-1\rangle}_{\delta_{n,n'-1}} + \underbrace{\langle n|n'+1\rangle}_{\delta_{n,n'+1}}$, which

yields the selection rule

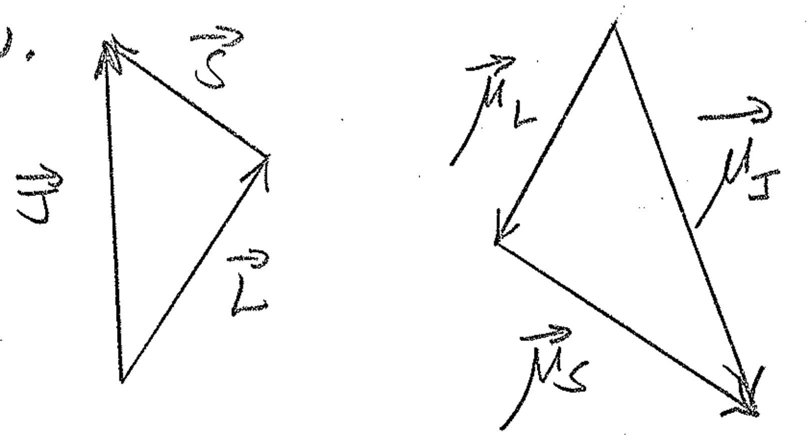
$n' = n \pm 1.$

#6. b) C-30, Spg, 1978

$\vec{\mu}_L = \frac{e}{2m}\vec{L}$, $\vec{\mu}_s = \frac{e}{m}\vec{S}$; $\vec{J} = \vec{L} + \vec{S}$

$\vec{\mu}_J = \vec{\mu}_L + \vec{\mu}_S = \frac{e}{2m}(\vec{L} + 2\vec{S})$; $\therefore$

$\vec{\mu}_J$ is not colinear with $\vec{J}$, as is shown below.



Within a given L-S multiplet the Zeeman splitting is different for different fine structure (or J-) levels. For a given J, M_J, the level shift is $\Delta E(J, M_J) \sim M_J g_J$, where g_J, the "Landé g-factor", is given by

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$