PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #29

January 14, 1978

General Instructions

This Comprehensive Examination consists of six problems of equal weight (20 points each). Half of the subject matter is judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

Some information you may find useful:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\quad, \text{ where } v^2 = T/\rho \quad .$$

$$\sum_{m_1, m_2} \langle m_1 m_2 | jm\rangle \langle m_1 m_2 | j'm'\rangle = \delta_{jj'}\delta_{mm'}$$

$$|jm\rangle = \sum_{m=m_1+m_2} |m_1 m_2\rangle \langle m_1 m_2 | jm\rangle$$

$$\psi_{n\ell m} = R_{n\ell}(r)\, \Theta_{\ell,m}(\theta)\, \Phi_m(\phi) \qquad \langle R_{n\ell} | R_{n'\ell'}\rangle = \delta_{nn'}\delta_{\ell\ell'}$$

$$\langle \Theta_{\ell'm'} | \Theta_{\ell m}\rangle = \delta_{\ell\ell'}\delta_{mm'} \qquad \langle \Phi_m | \Phi_{m'}\rangle = \delta_{mm'}$$

$$\langle \Theta_{\ell,m} | \cos\theta | \Theta_{\ell',m'}\rangle = \delta_{\ell,\ell'\pm 1}\, \delta_{|m|,|m'|}\, f(\ell,|m|)$$

$$h = 6.6\cdot 10^{-34} \text{ joule-sec} = 4.14\cdot 10^{-15} \text{ eV-sec}$$

$$a_0 = h^2/m_e e^2 = 0.529 \text{ Å}$$

$$E_0 = m_e e^4/2h^2 = 13.6 \text{ eV}$$

$$\psi_{100} = \pi^{-1/2}(a_0)^{-3/2}\exp(-r/a_0)$$

(20) 1. Determine the splitting of the $n = 3$ states of a spinless hydrogen atom, caused by an electric field in the z direction. Describe the number of energy levels, the degeneracy of each level, and the ℓ,m quantum numbers of the unperturbed states that go into each level.

(20) 2. Starting with the equations for the angular momentum raising and lowering operators

$$j_\pm|jm\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\,|j,m \pm 1\rangle$$

calculate the Clebsch-Gordon coefficients for the case where angular momenta $j_1 = 1$ and $j_2 = \frac{1}{2}$ are coupled together to make total angular momentum $j = \frac{1}{2}$.

3. A stretched uniform string of density ρ and length L is terminated at the end x = L by a ring of negligible mass which slides without friction on a vertical rod.

(5) (a) Determine the boundary condition at x = L.

(5) (b) If the end x = 0 is tied, or fixed, find the normal modes of vibration.

(5) (c) Determine the boundary condition at x = L if the ring is no longer massless, but has a mass M. Neglect gravity.

(5) (d) Set up the condition that determines the normal modes for part (c).

4. When a π^- meson (pion, $m = 273\ m_e$, $q = -e$) is trapped in the potential of an atom, it can form a hydrogen-like state if the charge Ze of the nucleus is not too large, in which its orbital radius is small compared to the radius of the electrons of the atom, and large compared to the size of the nucleus. In such a case, the potential experienced by the meson is $-Ze/r$. Since the wave-function does not overlap the nucleus strongly, the probability that the meson is within the nucleus is small, and there will be an appreciable time before the pion interacts with the nucleus, at which point there is a nuclear disintegration.

(8) (a) Evaluate the parameters describing the ground state of a mesonic atom formed by a $^7Li^3$ nucleus and a π^- meson.

(4) (b) Calculate the energy of radiation corresponding to a typical transition to the ground state of this mesonic atom.

(8) (c) Assuming that the nucleus is a sphere of radius 2.3×10^{-13} cm, calculate the probability that the ground state pion is in the nucleus at any instant, and estimate the life-time, i.e., the length of time in the ground state before reaction with the nucleus. (You may assume that if the pion enters the nucleus a nuclear reaction will occur.)

(10) 5. (a) A beam of particles is scattered by a fixed impenatrable sphere of radius R. The sphere is smooth, i.e., it can exert only a radial force. Determine the relation between a given impact parameter and the corresponding scattering angle. (Please justify all steps. Don't guess.)

(10) (b) A cloud chamber photograph shows the track of an incident particle which makes a collision and is scattered through an angle of 90^o. The track of the target particle makes an angle θ_2 with the direction of the incident particle. Assuming that the collision was elastic, and that the target particle was initially at rest, determine the ratio m_1/m_2 of the two masses.

(5) 6. (a) A particle is on a table in a uniform gravitational field. The energy is given by

$$E = mgz + \frac{p^2}{2m} ,$$

where z = 0 at the table. Classically the minimum energy is E = 0. Assume that the particle moves in a small range Δz above z = 0. Take the average height to be $\bar{z} = \Delta z/2$, and take $\Delta p \geq \hbar/2\Delta z$. Write $\bar{E}$ as a function of $\bar{z}$, and find the value $\bar{z}_{min}$ for which $\bar{E}$ is a minimum.

(10) (b) In part (a) the complete expression for the potential energy is given by

$$V(z) = mgz \text{ if } z > 0, \text{ and } V(z) = \infty \text{ if } z \leq 0.$$

Write out the time-independent Schroedinger equation for this problem; and determine the assymtotic forms of the wave function as $z \to \infty$ and as $z \to 0$.

(5) (c) Sketch the potential energy function described in part (b); and, on the same diagram, sketch the ground state wave function. Justify your sketch of the wave function.

$$H_{op}\psi = E\psi$$

where

$$H = H^0_{op} + V_{op} \;; \quad V = eE_z z = eE_z r\cos\theta$$

$$\psi \cong \sum_{l,m} c_{lm} |\psi^0_{3,l,m}\rangle$$

where $|\psi^0_{3,l,m}\rangle = R_{3l}(r)\,\Theta_{lm}(\theta)\,\Phi_m(\varphi)$

$$\sum_{lm} (H^0_{op} + V_{op} - E)\, c_{lm} |\psi^0_{3lm}\rangle = 0$$

Operate by $\langle\psi^0_{3l'm'}|$

$$\sum_{lm} \left[(E_0 - E)\,\delta_{ll'}\,\delta_{mm'} + V_{l'm'lm} \right] c_{lm} = 0 \quad \text{For each } l,m$$

Solution occurs for $\Delta E = E - E^0$ for which

$$\left| V_{l'm'lm} - \Delta E\,\delta_{ll'}\,\delta_{mm'} \right| = 0$$

$$V_{l'm'lm} = \langle \psi^0_{3l'm'} | V_{op} | \psi^0_{3lm} \rangle$$

$$= eE_z \langle R_{3l'} | r | R_{3l} \rangle \langle \Theta_{l'm'} | \cos\theta | \Theta_{lm} \rangle \langle \Phi_m | \Phi_{m'} \rangle$$

$$= eE_z \langle R_{3l'} | r | R_{3l} \rangle\, \delta_{m',m}\, \delta_{l'-1,l}\, f(l,|m|)$$

$$V_{l'm';lm} = 0 \quad \text{unless} \quad m' = m, \; l' = l \pm 1$$

As a result the secular equation Factors into the following equations

$m = \pm 2 \quad : \quad \Delta E = V_{22\,22} = 0$

$m = \pm 1 \quad \begin{vmatrix} -\Delta E & V_{21;11} \\ V_{11;21} & -\Delta E \end{vmatrix} = 0 \quad \text{ie} \quad \Delta E = \pm |V_{21;11}|$

$m = 0 \quad \begin{vmatrix} -\Delta E & V_{20;10} & 0 \\ V_{10;20} & -\Delta E & V_{10;00} \\ 0 & V_{00;10} & -\Delta E \end{vmatrix} = 0$

The last can be written

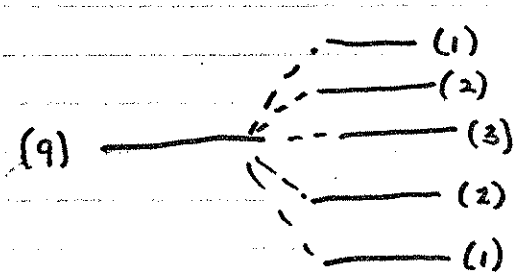
$$-\Delta E^2 + E\left[|V_{20;10}|^2 + |V_{10;00}|^2\right]$$

or

$$\Delta E = 0$$

$$\Delta E = \pm\sqrt{|V_{20;10}|^2 + |V_{10;00}|^2}$$

Therefore the splitting of the states is



Origin of levels

m	l
0	0, 1, 2
±1	1, 2
0, ±2	0, 1, 2
±1	1, 2
0	1, 2

#2

Soln: The C-G coefficients $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$ are defined by the expansion

$$|j\, m\rangle = \sum_{m=m_1+m_2} |m_1 m_2\rangle \langle m_1 m_2 | j\, m\rangle$$

The total ang. mom. raising & lowering operator $J_\pm = J_{1\pm} + J_{2\pm}$. Operate on above eqn. and multiply by $\langle m_1' m_2'|$.

$$\sqrt{(j\mp m)(j\pm m+1)}\ \langle m_1' m_2' | j, m\pm 1\rangle$$
$$= \sum_{m=m_1+m_2} \sqrt{(j_1\mp m_1)(j_1\pm m_1+1)}\langle m_1' m_2' | m_1\pm 1, m_2\rangle\langle m_1 m_2 | j\, m\rangle$$
$$+ \sum_{m=m_1+m_2} (j_2\mp m_2)(j_2\pm m_2+1)\langle m_1' m_2' | m_1, m_2\pm 1\rangle\langle m_1 m_2 | j\, m\rangle$$

Using the fact that $\langle m_1' m_2' | m_1 m_2\rangle = \delta_{m_1' m_1}\ \delta_{m_2' m_2}$ and dropping primes:

$$\sqrt{(j\mp m)(j\pm m+1)}\quad \langle m_1 m_2 | j, m\pm 1\rangle$$
$$= \sqrt{(j_1\pm m_1)(j_1\mp m_1+1)}\ \langle m_1\mp 1, m_2 | j\, m\rangle$$
$$+ \sqrt{(j_2\pm m_2)(j_2\mp m_2+1)}\ \langle m_1, m_2\mp 1 | j\, m\rangle$$

Since $m = m_1 + m_2$ there are 4 cases for which the C-G coefficient will not vanish

#2 Cont'd

m	m_1	m_2
$\frac{1}{2}$	1	$-1/2$
$\frac{1}{2}$	0	$1/2$
$-1/2$	-1	$1/2$
$-1/2$	0	$-1/2$

Substituting into the recursion relation yields

$$\langle -1\ 1/2\ |\ 1/2\ -1/2\rangle = \sqrt{2}\ \langle 0\ 1/2\ |\ 1/2\ 1/2\rangle$$

$$\langle 0\ -1/2\ |\ 1/2\ -1/2\rangle = \sqrt{2}\ \langle 1\ -1/2\ |\ 1/2\ 1/2\rangle + \langle 0\ 1/2\ |\ 1/2\ 1/2\rangle$$

$$\langle 1\ -1/2\ |\ 1/2\ 1/2\rangle = \sqrt{2}\ \langle 0\ -1/2\ |\ 1/2\ -1/2\rangle$$

$$\langle 0\ 1/2\ |\ 1/2\ 1/2\rangle = \sqrt{2}\ \langle -1\ 1/2\ |\ 1/2\ -1/2\rangle + \langle 0\ -1/2\ |\ 1/2\ -1/2\rangle$$

arbitrarily set $\langle 0, \frac{1}{2} | \frac{1}{2}, \frac{1}{2}\rangle \equiv c$ then

$$\langle -1\ 1/2\ |\ 1/2\ -1/2\rangle = \sqrt{2}\, c$$

$$\langle 0, -1/2\ |\ 1/2\ -1/2\rangle = -c$$

$$\langle 1, -1/2\ |\ 1/2, +1/2\rangle = -\sqrt{2}\, c$$

From the orthonormality of the C.G. coefficients

$$\sum_{m_1 m_2} \langle m_1\, m_2\ |\ j\, m\rangle^2 = 1$$

so $3c^2 = 1$. If we choose $c = -1/\sqrt{3}$

we get the C-G coefficients in the Condon and Shortley phase convention.

3(a) The total transverse force on the ring must vanish.

Hence $\left.\frac{\partial Y(x,t)}{\partial x}\right|_{x=L} = 0$

(b) Separation of variables yields the solution

$$Y(x,t) = \Theta(t)\, X(x) \quad \text{where}$$

$$\Theta = A \cos \omega t + B \sin \omega t$$

$$X = C \cos\left(\frac{\omega x}{v}\right) + D \sin\left(\frac{\omega x}{v}\right)$$

and the constants A, B, C, D, ω have to be determined.

Now $Y(0,t) = 0$ implies $C = 0$

and $\left.\frac{\partial X}{\partial t}\right|_{x=0} = 0$ implies $\cos\left(\frac{\omega L}{v}\right) = 0$

This fixes the normal modes $\omega_m = \frac{m\pi v}{2L}$ (m odd)

(c) If the ring is massive it will experience a finite acceleration $\left.\frac{\partial^2 Y}{\partial t^2}\right|_{x=L} = \frac{F_t}{m}$ where F_t is the transverse restoring force $T \sin\Theta \approx -T \left.\frac{\partial Y}{\partial x}\right|_{x=L}$

The B.C. is $\left.\frac{\partial^2 Y}{\partial t^2}\right|_{x=L} = -\frac{T}{m} \left.\frac{\partial Y}{\partial x}\right|_{x=L}$

(d) or $X(L)\ddot{\Theta} = -\frac{T}{m} \dot{X}(L)\, \Theta$ or $\omega^2 X(L) = \frac{T}{m} \dot{X}(L)$

or $\sin\left(\frac{\omega_n x}{v}\right) - \frac{T}{\omega_n m} \cos\left(\frac{\omega_n x}{v}\right) = 0$

This equation (if one could solve it) fixes the normal modes ω_n.

4 a

The ground state is hydrogen-like with $Z = 3$ and

$$\mu = \frac{m_\pi m_{Li}}{m_\pi + m_{Li}} = \frac{m_\pi}{1 + \frac{m_\pi}{m_{Li}}} = \frac{273\, m_e}{1 + \frac{273}{7\times 1836}} = 267.3\, m_e$$

$$\therefore E_0 = \frac{\mu_\pi}{m_e} Z^2 \frac{m_e e^4}{2\hbar^2} = 267.3\times 9\, E_0 = 3.27\times 10^4\, eV$$

$$a_\pi = \frac{\hbar^2}{\mu_\pi Z e^2} = \frac{m_e}{m_\pi Z}\times a_0 = 6.60\times 10^{-12}\ cm$$

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{a_\pi^{3/2}}\, e^{-r/a_\pi}$$

b) If the transition is from $n=2$ to $n=1$

$$h\nu = E(n=2) - E(n=1) = -\frac{E_0}{4} + \frac{E_0}{1} = \frac{3E_0}{4} = 2.45\times 10^4\, eV$$

c)

$$P = \int_{\substack{\text{volume}\\ \text{of nucleus}}} |\psi|^2 d\tau = \frac{1}{\pi a_\pi^3}\int_0^R e^{-2r/a_\pi}\, 4\pi r^2 dr$$

Since $\frac{2r}{a_\pi} < \frac{2R}{a_\pi} \ll 1$

$$e^{-2r/a_\pi} \cong 1$$

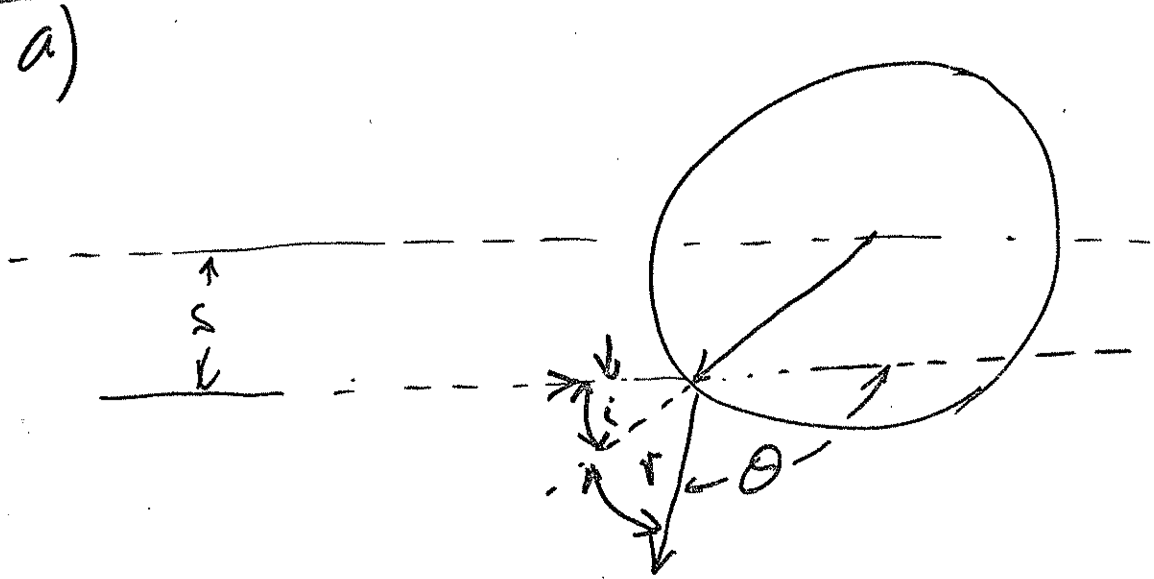
$$P \cong \frac{1}{\pi a_\pi^3}\times\frac{4\pi}{3}R^3 = 5.64\times 10^{-5}$$

The lifetime $\tau \cong \frac{1}{\text{frequency of motion}\times P} = \frac{h}{|E_{100}|\, P}$

$$= 2.24\times 10^{-15}\ sec$$

Mech. Solution

#5 a)



(2)

A particle incident from the left with impact parameter s is scattered through an angle θ. To determine θ, note that during impact the tangential component of momentum is constant - only the radial component changes. Since the collision is elastic the magnitude of the momentum remains constant. (3) Thus we have

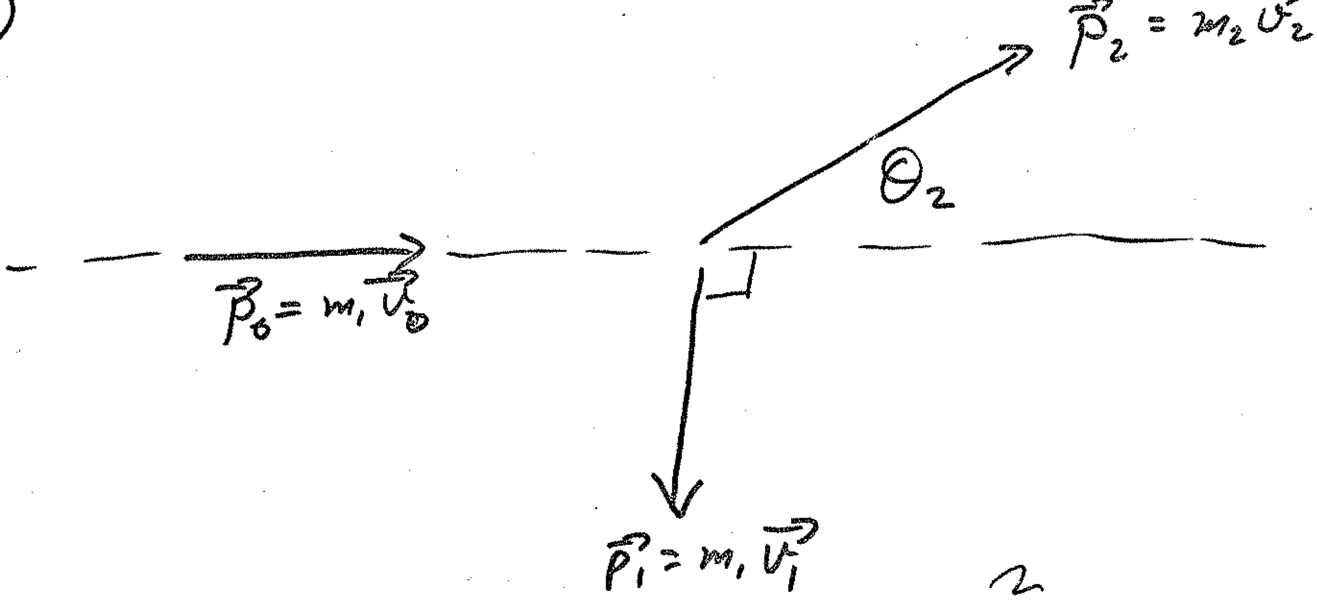
(2)

In other words, in order that $|\vec{p}_0|$ and $|\vec{P}_{0t}|$ be constant, $\vec{P}_{0r}$ must be reversed by the collision. Thus, $i = r$, & $2i = \pi - \theta$. But $s = R\sin i$;

$\therefore$, $s = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cos\frac{\theta}{2}$ (3)

#5 ~~Grad~~ Mech Solution

b)



$$m_1 v_0 = m_2 v_2 \cos\theta_2 \,, \quad m_1 v_1 = m_2 v_2 \sin\theta_2$$

Since the collision is elastic $m_1 v_0^2 = m_1 v_1^2 + m_2 v_2^2$ 6

Thus, $m_1 v_1^2 + m_2 v_2^2 = \frac{m_2^2}{m_1} v_2^2 \cos^2\theta_2$, or

$$\frac{m_2^2}{m_1} v_2^2 \sin^2\theta_2 + m_2 v_2^2 = \frac{m_2^2}{m_1} v_2^2 \cos^2\theta_2 \,, \text{ or}$$

$$\frac{m_2}{m_1} \sin^2\theta_2 + 1 = \frac{m_2}{m_1} \cos^2\theta_2 \,, \text{ or}$$

$$\frac{m_1}{m_2} = \cos^2\theta_2 - \sin^2\theta_2 = 1 - 2\sin^2\theta_2$$

4

#6 U.G QM Solution

a) Assume $\bar{p} \simeq \Delta p/2 \simeq \hbar/4\Delta z \simeq \frac{\hbar}{8\bar{z}}$.

Then $\bar{E} \simeq mg\bar{z} + \frac{\hbar^2}{128m(\bar{z})^2}$; and

$$\frac{\partial \bar{E}}{\partial \bar{z}} = mg - \frac{\hbar^2}{4m(\bar{z})^3} = 0 \text{ if } (\bar{z})^3 = \frac{\hbar^2}{4m^2 g};$$

$$\therefore \quad \bar{z}_{min} = \left(\frac{\hbar^2}{4m^2 g}\right)^{1/3}.$$

b) 4

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(z)}{dz^2} + (mgz - E)\psi(z) = 0 \text{ if } z > 0,$$

$$\psi = 0 \text{ if } z \leq 0.$$

3

Since $\psi(0) = 0$ $\left.\frac{d^2\psi}{dz^2}\right|_0 = 0$, or $\left.\frac{d\psi}{dz}\right|_0 = $ constant.

Thus, near the origin, $\psi(z) \sim z$.

3

As $z \to \infty$ we have $\psi'' - \alpha z \psi = 0$. If z were constant, this would give $\psi \sim e^{-(\sqrt{\alpha z})z}$;

$\therefore$, assume $\psi \sim e^{-cz^{3/2}}$. $\psi' \sim -\frac{3}{2}cz^{1/2}e^{-cz^{3/2}}$

$$\psi'' \sim \left[-\frac{3}{4}cz^{-1/2} + \frac{9}{4}c^2 z\right]e^{-cz^{3/2}}.$$

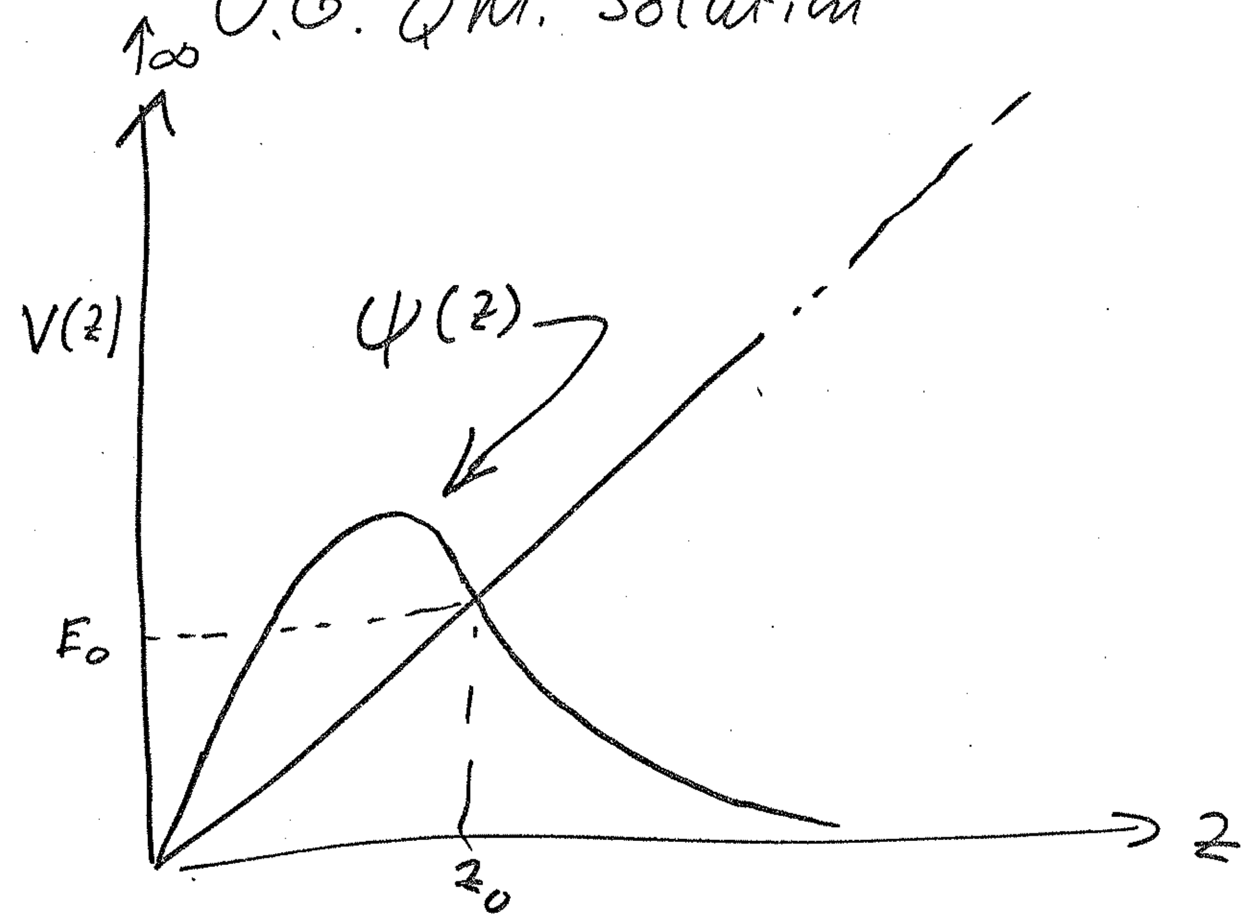
$$-\frac{3}{4}cz^{-1/2} + \frac{9}{4}c^2 z - \alpha z = 0, \text{ which is O.K.}$$

$$\text{if } c = \frac{2}{3}\sqrt{\alpha}$$

10

U.G. QM. Solution

#6 c)



In the classically forbidden region beyond $z = z_0$ the wave function decays exponentially, as seen in part (b); and it rises linearly near $z = 0$. The ground state wv f'n must have the least amount of oscillation - or the fewest maxima & minima; ∴ we have only one maxima, with $0 < z_{max} < z_0$.