PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #28

October 8, 1977

General Instructions

This Comprehensive Examination for Fall 1977 (#28) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

Some information you may find useful:

$$\int \sin^2 x dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) = \frac{1}{2}x - \frac{1}{2}\cos x \sin x$$

$$\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{2}\sin x \cos x$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

$$\rho = \frac{1 - n}{1 + n} \qquad \left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}x$$

$h = 6.6 \times 10^{-34}$ joule-sec

$c = 3 \times 10^{8}$ m/sec

$a_0 = 0.5 \times 10^{-10}$ m

$k = 1.38 \times 10^{-23}$ joule/°K

(20) 1. A uniform thin film of transparent liquid (soap film) with thickness d and refractive index n causes interference in transmitted light due to multiple reflections within the film. For white light at normal incidence, derive an expression for the fraction T of transmitted light at a given wavelength λ, and determine the ratio T_{max}/T_{min} for adjoining transmission maxima and minima, assuming that n is independent of λ.

(15) 2. (a) A projectile is fired horizontally along the earth's surface. Show that to a first approximation the angular deviation from the direction of fire resulting from the Coriolis force varies linearly with time at a rate $\omega \cos \theta$, where ω is the angular frequency of the earth's rotation and θ is the colatitude.

(5) (b) Derive the direction of the Coliolis force as seen in the northern hemisphere.

3. The two lowest single-particle eigenfunctions of the infinitely deep one-dimensional square well problem are given by

$$\psi_1(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right) \quad \text{and} \quad \psi_2(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right),$$

where $-a \leq x \leq a$. These eigenfunctions are normalized.

(5) (a) Determine the smallest single-particle energy eigenvalue for this problem.

(5) (b) Normalize the following single-particle eigenfunction:

$$\psi(x) = \psi_1(x) + 2\psi_2(x).$$

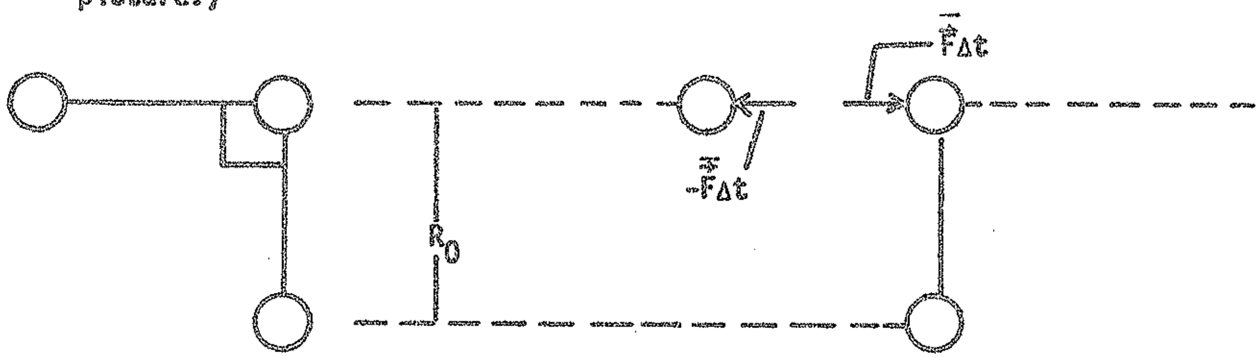
(5) (c) Determine the selection rule(s) for the one-particle dipole transition matrix elements for this problem.

(5) (d) Consider two non-interacting fermions bound in an infinitely deep one-dimensional square well. Write out the expression for the space part of the lowest energy eigenfunction for this system which has odd parity and is antisymmetric with respect to exchange of particles.

4. Suppose that in the simple photodissociation of a triatomic molecule the dissociation process can be treated as the breakage, or rupture, of a chemical bond. Thus, for example, the process

$$\gamma + O_3 \rightarrow O_2 + O \;,$$

which dominates the sunlight photodestruction of ozone in the stratosphere, could be depicted as shown below. (In fact the bond angle for O_3 is 117° rather than 90°; however, this represents a minor correction of the picture.)



The impulse, $|\vec{F}\Delta t|$, given each photofragment during dissociation is assumed to occur entirely along the line shown. This impulse gives rise to released energy (RE), which is partitioned between released kinetic energy (RKE) and internal excitation of the O_2 fragment.

(10) (a) What fraction of RKE is carried away by the O-atom?

(10) (b) Assume the O_2 molecule to be a classical rigid rotator, and determine the relation between RKE and the O_2 rotational energy, E_R. Also, determine the ratio RE/RKE.

5. A system contains N fixed atoms per unit volume (i.e., in a crystal) which are non-interacting and each atom gives rise to four two-electron states. Each atomic state is described by an occupancy number $n = 0, 1,$ or 2, and a total spin which, because of the Pauli principle, is non-zero only if $n = 1$. The energies of the possible states of each atom in a magnetic field B are given in the table below:

n	s	E(n,s)
0	0	0
1	+1/2	$-T-\mu_b B$
1	-1/2	$-T+\mu_b B$
2	0	$-2T+U$

In the table T is the binding energy of an electron, and U is the repulsion energy for two electrons in the same one-electron state.

(5) (a) Determine the grand partition function $Z(B,\beta,\mu)$ for this system and express it in a closed form. Remember that $\beta = 1/kt$, and μ is the chemical potential of an electron (Fermi energy). Hint: For N independent particles,

$$Z = \prod_{j}^{N} \sum_{n_j, s_j} e^{-\beta[E(n_j,s_j) - \mu n_j]} \ .$$

(5) (b) What is the magnetic moment M of the system?

(5) (c) What relation must be satisfied if there is spontaneous magnetization?

(5) (d) Derive a relation which fixes the value of μ, given the average number of electrons per atom is f, and also given β and B.

6. Suppose one were to build a radio telescope consisting of N small, equally spaced antennas mounted in a straight line.

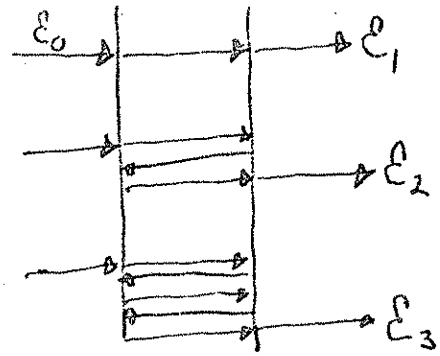
(5) (a) Assume that the spacing of the individual antennas is d and the wavelength of the source is λ. Find the direction(s) of maximum gain.

(5) (b) Find the power gain of this array as compared with a single antenna.

(10) (c) Derive the angular resolution function (sensitivity function) for this apparatus.

#1

1



$$\mathcal{E}_1 = \mathcal{E}_0 e^{i\phi}(1-\rho)(1+\rho) = \mathcal{E}_0 e^{i\phi}(1-\rho^2)$$

$$\rho = \frac{1-n}{1+n}$$

$$\phi = \frac{2\pi n d}{\lambda}$$

$$\mathcal{E}_2 = \mathcal{E}_1 \rho^2 e^{2i\phi}$$

$$\vdots$$

$$\mathcal{E}_n = \mathcal{E}_1 \left(\rho^2 e^{2i\phi}\right)^{n-1}$$

$$\mathcal{E} = \sum_{n=1}^{\infty} \mathcal{E}_n = \mathcal{E}_1 \sum_{n=0}^{\infty} \left(\rho^2 e^{2i\phi}\right)^n$$

$$\mathcal{E} = \frac{\mathcal{E}_1}{1-\rho^2 e^{2i\phi}} = \frac{\mathcal{E}_1}{1-\rho^2\cos 2\phi - i\rho^2 \sin 2\phi}$$

$$\mathcal{I} = |\mathcal{E}|^2 = \frac{\mathcal{E}_1^{\,2}}{(1-\rho^2\cos 2\phi)^2 + \rho^4 \sin^2 2\phi} = \frac{\mathcal{E}_0^{\,2}(1-\rho^2)^2}{(1-\rho^2\cos 2\phi)^2 + \rho^4 \sin^2 2\phi}$$

$$T = \frac{\mathcal{I}}{|\mathcal{E}_0|^2} = \frac{(1-\rho^2)^2}{(1-\rho^2\cos 2\phi)^2 + \rho^4 \sin^2 2\phi} = \frac{(1-\rho^2)^2}{1-2\rho^2\cos 2\phi + \rho^4}$$

When $\cos 2\phi = 1$, $T = T_{max} = \frac{(1-\rho^2)^2}{1-2\rho^2+\rho^4} = 1$

When $\cos 2\phi = -1$ $T = T_{min} = \frac{(1-\rho^2)^2}{1+2\rho^2+\rho^4} = \left(\frac{1-\rho^2}{1+\rho^2}\right)^2$

$$\frac{T_{max}}{T_{min}} = \frac{(1+\rho^2)^2}{(1-\rho^2)^2} = \frac{\left[(1+n)^2+(1-n)^2\right]^2}{\left[(1+n)^2-(1-n)^2\right]^2} = \left[\frac{2+2n^2}{4n}\right]^2 = \frac{(1+n^2)^2}{4n^2}$$

#2

— Graduate Mechanics, Soln. —

Let $\vec{r}$ be the position of the projectile with respect to the center of the earth. We need the first and second time derivative as seen from a rotating coordinate system (i.e. the earth)

$$\left.\frac{d\vec{r}}{dt}\right)_s = \left.\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\vec{r}$$

$$\left.\frac{d^2\vec{r}}{dt^2}\right)_s = \left.\frac{d^2\vec{r}}{dt^2}\right)_r + 2(\vec{\omega}\times\vec{v}_r) + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

The second term on the right is the Coriolis acceleration. For a horizontal projectile the hor. component of the acceleration is $2\omega v\cos\theta$. After a time t the linear deflection is $\omega t^2 v\cos\theta$ while the projectile has covered a distance vt. Hence the angular displacement

$$\theta \approx \frac{\omega t^2 v\cos\theta}{vt} = \omega t\cos\theta$$

The Coriolis "force" in the rotating c.s. is $-2m(\vec{\omega}\times\vec{v})$. The earth rotates c.c.w. as viewed from the north pole hence the force is to the right in the northern hemisphere.

#3 U. G Quantum Mech - sol'n.

a) $H\psi_1 = E_1\psi_1$ $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$; $\therefore E_1\psi_1 = \frac{\hbar^2}{2m}\cdot\frac{\pi^2}{4a^2}\psi_1$,

or $E_1 = \frac{\pi^2\hbar^2}{8ma^2}$.

b) $$1 = \int_{-a}^{a}\psi^*\psi\,dx = \frac{c^*c}{a}\Big[\underbrace{\int_{-a}^{a}\cos^2\left(\frac{\pi x}{2a}\right)dx}_{\mathcal{J}_1} + 4\underbrace{\int_{-a}^{a}\sin^2\left(\frac{\pi x}{a}\right)dx}_{\mathcal{J}_2}\Big]$$

$$\mathcal{J}_1 = 2\cdot\frac{2a}{\pi}\int_0^{\frac{\pi}{2}}\cos^2u\,du = a \cdot \mathcal{J}_2 = 2\cdot\frac{a}{\pi}\int_0^{\pi}\sin^2u\,du = a \;;$$

$\therefore$, $1 = \frac{c^*c}{a}[a + 4a]$, or $c^2 = \frac{1}{5}$ & $c = \frac{1}{\sqrt{5}}$.

c) $\langle n|x|n'\rangle = 0$ ~~unless~~ if n, n' are both odd or both even; $\therefore$ the sel'n rule is that $\Delta n = |n'-n|$ be odd

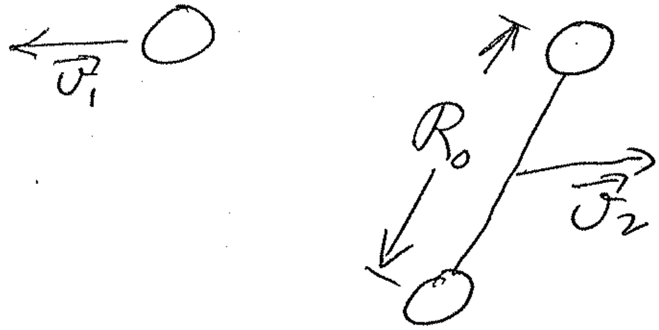
d) $\psi^0_{as}(x_a, x_b) =$ ~~$\frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)]$, where~~

$$= \frac{1}{\sqrt{2}}[\psi_1(x_a)\psi_2(x_b) - \psi_2(x_a)\psi_1(x_b)],$$

where the subscripts 1, 2 refer to the s.p. wv f'ns defined above and a, b distinguish the two fermions.

#4 U.G. mechanics – Sol'n

a)



$$m v_1 = 2m v_2 \;. \quad RKE = \tfrac{1}{2} m v_1^2 + \tfrac{1}{2}(2m) v_2^2 \;; \therefore$$

$$RKE = (KE)_0 + \frac{m v_1^2}{4} = \frac{3}{2}(KE)_0 \;;$$

or, $(KE)_0 = \frac{2}{3} RKE$

b) $$E_R = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2}\left[2m\left(\frac{R_0}{2}\right)^2\right]\dot{\theta}^2,$$

where $$\dot{\theta} = \frac{2 v_2}{R_0} \;; \therefore, \; E_R = m v_2^2 = \tfrac{1}{2}\left(\tfrac{1}{2} m v_1^2\right);$$

or $E_R = \frac{1}{2}(KE)_0 = \frac{1}{3} RKE.$

$$RE = RKE + E_R = RKE + \frac{1}{3} RKE,$$

or $RE / RKE = 4/3$.

5 a

$$Z = \prod_{j=1}^{N} \left[\sum_{n_j s_j} e^{-\beta E(n_j, s_j) + \beta n_j \mu} \right]$$

$$= \prod_{j=1}^{N} \left[1 + e^{\beta(T+\mu_B B+\mu)} + e^{\beta(T-\mu_B B+\mu)} + e^{-\beta U} e^{2\beta(T+\mu)} \right]$$

$$= \left[1 + 2e^{\beta(T+\mu)} \cosh \beta\mu_B B + e^{-\beta U} e^{2\beta(T+\mu)} \right]^N$$

b

$$M = \frac{1}{\beta} \frac{\partial \ln Z}{\partial B} = \frac{N\mu_B \times 2 \sinh \mu_B B\, e^{\beta(T+\mu)}}{1 + 2e^{\beta(T+\mu)} \cosh \mu_B B + e^{-\beta U} e^{2\beta(T+\mu)}}$$

c) If there is spontaneous magnetization

$$B = \mu_0 M$$

$$\therefore \frac{B}{N\mu_B \mu_0} = \frac{2 \sinh \mu_B B\, e^{\beta(T+\mu)}}{1 + 2e^{\beta(T+\mu)} \cosh \mu_B B + e^{-\beta U} e^{2\beta(T+\mu)}}$$

d)

$$n = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \mu}$$

$$f = \frac{n}{N} = \frac{2\left[\cosh \mu_B B\, e^{\beta(T+\mu)} + e^{-\beta U} e^{2\beta(T+\mu)} \right]}{1 + 2e^{\beta(T+\mu)} \cosh \mu_B B + e^{-\beta U} e^{2\beta(T+\mu)}}$$

6

Solution for Graduate optics problem:

The sum of the signals from the individual antennas is given by the real part of

$$E = E_0(r)\left[e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)} + \cdots e^{i(kr_N-\omega t)}\right]$$

$$= E_0\, e^{i(kr_1-\omega t)}\left[1 + e^{ik(r_2-r_1)} + e^{ik(r_3-r_1)} + \cdots + e^{ik(r_N-r_1)}\right]$$

assuming that the source is at infinity

$$k(r_j - r_1) = (j-1)\,kd\sin\theta \equiv (j-1)\,\delta$$

Compared with a single antenna

$$\frac{E}{E_1} = 1 + e^{i\delta} + \cdots + e^{i(N-1)\delta}$$

$$= \frac{e^{i\delta N}-1}{e^{i\delta}-1} = \frac{e^{iN\delta/2}}{e^{i\delta/2}}\,\frac{\left[e^{iN\delta/2}-e^{-iN\delta/2}\right]}{\left[e^{i\delta/2}-e^{-i\delta/2}\right]}$$

$$= e^{i(N-1)\delta/2}\;\frac{\sin N\delta/2}{\sin\delta/2}$$

The power gain

$$\frac{I}{I_1} = \left|\frac{E}{E_1}\right|^2 = \frac{\sin^2 N\delta/2}{\sin^2\delta/2}$$

This is maximum when $\delta/2 = m\pi$; $m = 0, \pm1, \pm2, \cdots$

or $\sin\theta_m = 2m\pi/kd$

(a) $d\sin\theta_m = m\lambda$

#8 (cont'd)

At these values

(b) $\frac{I}{I_1} = \lim_{\delta \to 2m\pi} \left(\frac{\sin^2 N\delta/2}{\sin^2 \delta/2} \right) = N^2$

(c) $I = I_0 \frac{\sin^2 [(Nkd/2) \sin\theta]}{\sin^2 [(kd/2) \sin\theta]}$