PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #27

April 2, 1977

General Instructions

This Comprehensive Examination for Spring 1977 (#27) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

Some information you may find useful:

$$h = 6.6 \times 10^{-34} \text{ joule-sec}$$
$$c = 3 \times 10^{8} \text{ m/sec}$$
$$a_0 = 0.5 \times 10^{-10} \text{ m}$$
$$k = 1.38 \times 10^{-23} \text{ joule/°K}$$
$$e = 1.6 \times 10^{-19} \text{ coul}$$

1\. (a) Deduce the selection rules for electric dipole transitions by an
(8) electron in an infinitely deep one-dimensional square well.
(Hint: Explicit evaluation of integrals is not necessary.)

(b) Consider an infinitely deep one-dimensional square well which extends from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. A particle in this well is perturbed by the potential: $V = 0$ if $x < 0$

(8) $$V = \frac{3\pi\hbar^2}{160mL^2}\sin\frac{\pi x}{L} \quad \text{if} \quad x > 0.$$

Determine the fractional energy change, caused by this perturbation, of the ground state of the system. (m = mass of particle)

(c) Suppose atoms were made up of electrons bound in one-dimensional
(4) potentials. Discuss the shell structure and thus the resulting periodic table for these atoms.

2\. The Ω^- particle is produced by the collision between negative K-mesons and protons, according to the reaction

$$K^- + p \rightarrow K^+ + K^0 + \Omega^-$$

The rest masses are: $m_{K^\pm}c^2 = 494$ MeV; $m_{K^0}c^2 = 498$ MeV; $m_p c^2 = 938$ MeV; $m_\Omega c^2 = 1672$ MeV.

(a) Compute the threshold lab kinetic energy for production of Ω^- when
(15) a beam of K^- mesons is incident on a target of protons at rest in the laboratory.

(b) Assume that we have selected Ω^- particles with a kinetic energy of
(5) 1000 MeV. The Ω^- is unstable with a lifetime of 1.1×10^{-10} sec. How long a track will these particles leave in the laboratory detection apparatus?

3. Imagine a semi-infinite opaque screen containing thirty identical parallel slits which are infinite in length and of width b. The distance between adjacent slits is irregular and random. Each slit is illuminated separately
(20) by a monochromatic plane wave of wavelength λ incident normally upon it. The separate incident waves are completely incoherent with respect to each other. Derive the expression for the Fraunhofer diffraction pattern produced by the light passing through the slits as observed on the other side of the screen.

4. (a) Using classical thermodynamics, derive Stefan's law for the energy
(15) density of electromagnetic radiation in a radiant cavity. (Hint: Begin with the first law and the definition of entropy.)

(b) Derive Stefan's law from the statistical mechanics of a photon gas,
([illegible]) assuming the photons to be bosons.

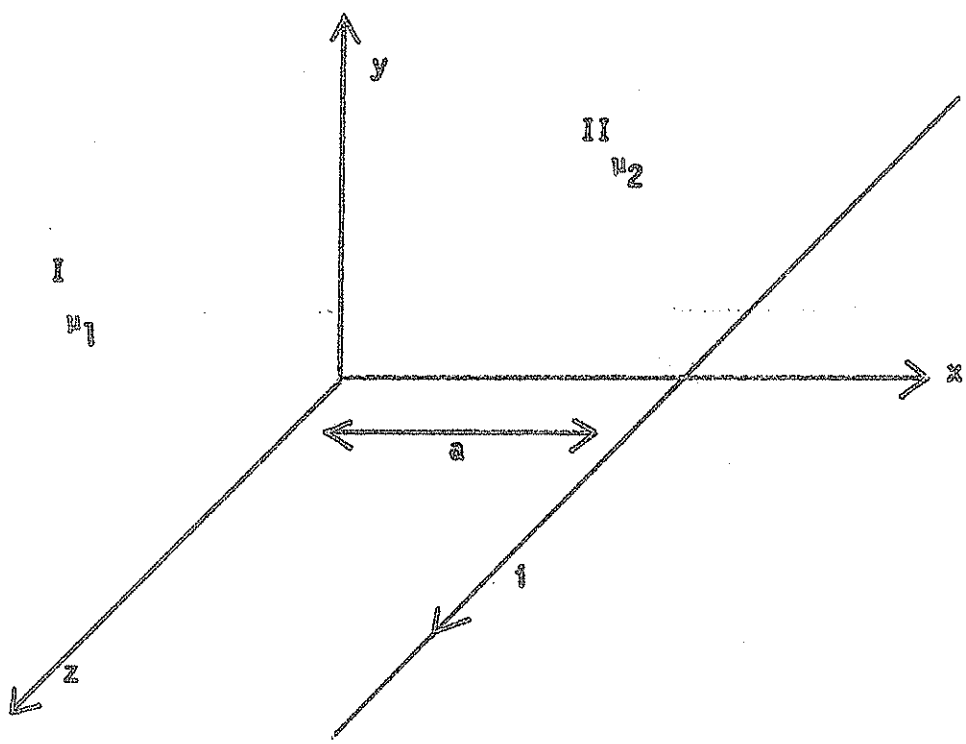
Note: In (a) or (b) you need not evaluate the constant of proportionality in Stefan's law; merely show the temperature dependence.

5. An infinitely long straight wire carries a current "i" in the +z direction. It passes through the xy plane at x=a, y=0. The region x < 0 (Region I) is filled with material of permeability μ_1. The region x > 0 (II) is filled with material of permeability μ_2. (Hint: The magnetic fields can be expressed in terms of image currents.)

(4) (a) What are the relevant boundary conditions which must be satisfied at the yz plane?

(12) (b) What are the magnitude and location of the required image currents?

(4) (c) In what direction is there a force on the wire?



6. We are interested in studying the rotational energy levels of a free diatomic molecule. For the purpose of this problem, ignore the dynamic behavior of the electrons, and assume the electronic energy is included in an unspecified potential function which depends only on the separation of the two nuclei.

(4) (a) Start by writing down the quantum mechanical Hamiltonian which describes this diatomic molecule, and then express it in a form which separates the center of mass motion from the relative motion. (You need not derive all the steps, but do define non-obvious terms.)

(8) (b) Separate the relative Hamiltonian of part (a) into radial and angular parts and determine the rotational eigenvalue equation for the case in which the molecule is treated as a rigid rod which rotates about the center of mass.

(4) (c) Calculate the difference in energy between the lowest two rotational levels in the hydrogen molecule, H_2. (The observed separation of the H atoms is 0.74 Angstroms.)

(4) (d) By comparing your answer of (c) with kT, estimate if the lowest level is excited at room temperature.

(1)

b) $\psi_{00} = A\cos\frac{\pi x}{L}$, where $\frac{1}{A^2} = \int_{-\frac{L}{2}}^{L/2} \cos^2\frac{\pi x}{2}\,dx = \frac{L}{2}$

$$\Delta E = \int \psi_{00}^{*} H_1 \psi_{00}\,dx = \frac{2}{L}\cdot\frac{3\pi h^2}{160 mL^2}\underbrace{\int_{-0}^{\frac{L}{2}} \sin\frac{\pi x}{2}\cos^2\frac{\pi x}{L}\,dx}_{\mathcal{J}}$$

$$\mathcal{J} = \int_{0}^{\frac{\pi}{2}} \sin\theta\cos^2\theta\,d\theta\cdot\frac{L}{\pi} = -\frac{L}{3\pi}\cos^3\theta\Big]_0^{\pi/2} = \frac{L}{3\pi}\ ;\ \therefore$$

$\Delta E = \frac{h^2}{80 mL^2}$. To get E_0, use $L = \frac{\lambda}{2}$, $2L = \frac{h}{p}$,

$$\frac{p^2}{2m} = E_0 = \frac{h^2}{2m\cdot 4L^2} = \frac{h^2}{8mL^2}\ ;\ \therefore \frac{\Delta E}{E_0} = 10^{-1}.$$

a) The w.v f'ns are alternately even and odd in x; and x is an odd function. The transition edm is $\underline{X} \propto \int_{-\frac{L}{2}}^{L/2} \psi_f^{*}\, x\, \psi_i\,dx$; $\therefore$, ψ_f and ψ_i must be of opposite parity, and the selection rule is $\Delta n = 1, 3, 5, \ldots$, where n is the E-level qu. no.

c) Two columns, each with approximately 50 entries.

[illegible]

2.(a) C of M frame: $\xrightarrow{p_K'}$ $\xleftarrow{p_p'}$

K p

$$E_i' = \sqrt{p_K'^2c^2 + m_K^2c^4} + \sqrt{p_p'^2c^2 + m_p^2c^4}$$

$$E_f' = m_{K^+}c^2 + m_{K^0}c^2 + m_{\Omega}c^2 = 2664 \text{ MeV}$$

$$p_K' = p_f'$$

$$(p_K'c)^2 + m_K^2c^4 = \left[(2664) - \sqrt{p_p'^2c^4 + m_p^2c^4}\right]^2$$

$$\cancel{(p_K'c)^2} + (494)^2 = (2664)^2 + \cancel{p_p'^2c^4} + (938)^2 - 2(2664)\sqrt{p_p'^2c^2 + m_p^2c^4}$$

$$p_p'^2c^2 + (938)^2 = \left[\frac{(2664)^2 + (938)^2 - (494)^2}{2(2664)}\right]^2$$

$$p_p' = 1107 \text{ MeV}/c = p_K'$$

$$cp' = m_0c^2\beta'/\sqrt{1-\beta'^2}$$

$$\beta_p' = 0.763c \qquad \beta_K' = 0.913c$$

Transform to lab frame (in which p is at rest)

$$\beta_K = \frac{\beta_p' + \beta_K'}{1 + \beta_p\beta_K} = 0.9878c$$

$$\gamma_K = 6.433 \qquad E_K = 3178 \text{ MeV}$$

$$\boxed{K_K = 2684 \text{ MeV}}$$

[illegible]

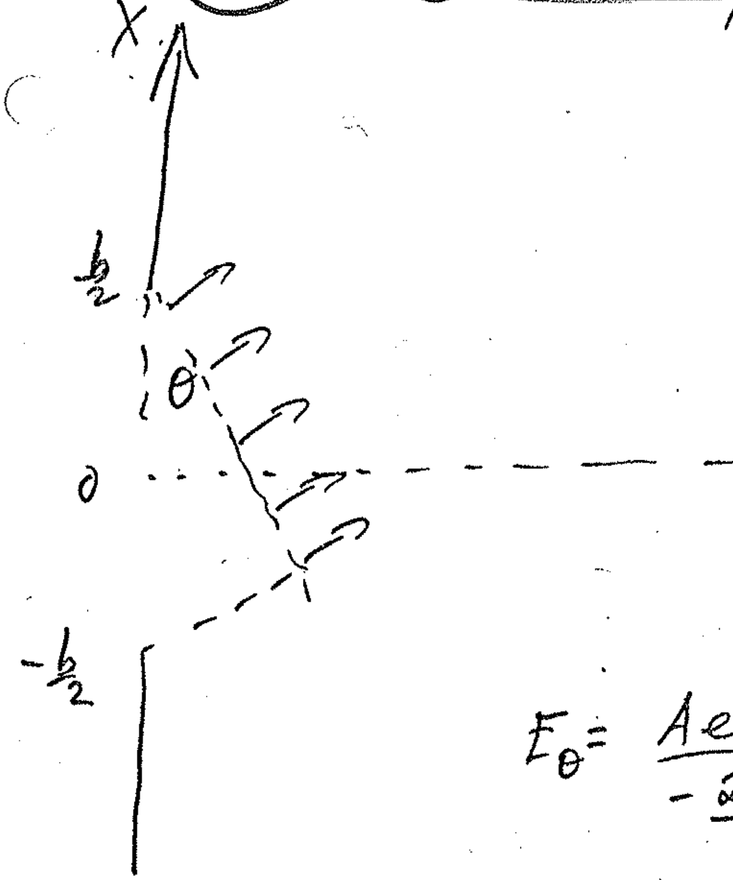
(b) Lab frame: $E = K + mc^2 = 2672 = \gamma(1672)$

$$\gamma = 1.598 \quad \Rightarrow \quad \beta = 0.78$$

$$T_{Lab} = \gamma T_0 = (1.6)(1.1\times10^{-10}) = 1.8\times10^{-10}\ \text{sec}$$

$$l = \beta c t = (0.78)(3\times10^{10})(1.8\times10^{-10}) = \underline{4.2\ \text{cm}}$$

3



$$dE_\theta(x) \propto dx\, e^{i[\omega t - \alpha_\theta(x)]},$$

where $\alpha_\theta(x) = \frac{2\pi}{\lambda}\Delta_\theta(x)$, and

$\Delta_\theta(x) = x\sin\theta$. Thus,

$$E_\theta \propto e^{i\omega t}\int_{-b/2}^{b/2} e^{-\frac{2\pi i}{\lambda}x\sin\theta}\,dx \quad , \text{ or,}$$

$$E_\theta = \frac{A e^{i\omega t}}{-\frac{2\pi i}{\lambda}\sin\theta}\left[e^{-\frac{\pi i}{\lambda}b\sin\theta} - e^{\frac{\pi i}{\lambda}b\sin\theta}\right]$$

$$= A e^{i\omega t}\,\frac{\sin\gamma}{\gamma}, \text{ where } \gamma = \frac{\pi b\sin\theta}{\lambda}$$

$$I/I_0 = \left(\frac{\sin\gamma}{\gamma}\right)^2$$. Now multiply by thirty.

(4.)

(a) $$T\,dS = T\left(\frac{\partial S}{\partial T}\right)_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV = dU + P\,dV$$

u = energy density of cavity.

$U = uV \qquad dU = V\,du + u\,dV$

classical atomic gas: $P = \frac{2}{3}u$

photon gas $\quad P = \frac{1}{3}u$

$$\therefore T\left(\frac{\partial S}{\partial T}\right)_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV = V\,du + u\,dV + \frac{u}{3}\,dV$$

$$T\left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}u$$

$$T\left(\frac{\partial S}{\partial T}\right)_V dT = V\,du = V\frac{du}{dT}\,dT$$

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{du}{dT}$$

$$\left[\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T\right]_V = \left[\frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V\right]_T$$

$$\frac{\partial}{\partial T}\left(\frac{4}{3}\frac{u}{T}\right) = \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{du}{dT}\right)$$

$$\frac{4}{3}\left(-\frac{u}{T^2}\right) + \frac{4}{3T}\frac{du}{dT} = \frac{1}{T}\frac{du}{dT} \Rightarrow \frac{4}{3}\frac{u}{T^2} = \frac{1}{3T}\frac{du}{dT}$$

$$4\frac{dT}{T} = \frac{du}{u} \Rightarrow \ln u = 4\ln T \Rightarrow \boxed{u = \sigma T^4}$$

(b) $f = \dfrac{1}{e^{h\nu/kT} - 1}$

density of states $\propto \nu^2 d\nu$

$$dn \propto \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1}$$

energy density $\propto \int h\nu \, dn$

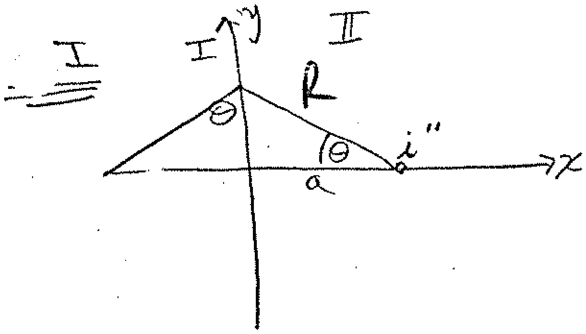
$$u \propto \int \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

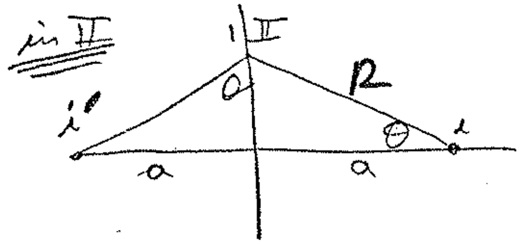
$$x = \frac{h\nu}{kT} \qquad dx = \frac{h}{kT} d\nu$$

$$u \propto T^4 \int \frac{x^3 dx}{e^x - 1} = \sigma T^4$$

⑤

E.+M. Solution





ⓐ tangential H, normal B continuous

$H_{1y} = H_{2y}$ $\qquad B_{1x} = B_{2x}$ $\qquad B = 2I/cR$

$\mu_1 H_{1x} = \mu_2 H_{2x}$

ⓒ $H_{1y} = -i'' \cos\theta\,(2/cR)$

$B_{1x} = \mu_1 H_{1x} = -\mu_1 i'' \sin\theta\,(2/cR)$ (not μ_2)

$H_{2y} = -i\cos\theta + i'\cos\theta\ (2/cR)$

$H_{2x} = -i\sin\theta - i'\sin\theta\ (2/cR)$

$B_{2x} = \mu_2 H_{2x} = -i\mu_2 \sin\theta - i'\mu_2\sin\theta\,(2/cR)$

$H_{1y} = H_{2y} \Rightarrow -i'' = -i + i' \Rightarrow i = i' + i''$

$B_{1x} = B_{2x} \Rightarrow -\mu_1 i'' = -i\mu_2 - i'\mu_2 \qquad i = \frac{\mu_1}{\mu_2} i'' - i'$

add $\quad 2i = \frac{\mu_1 + \mu_2}{\mu_2} i''$

$$\boxed{i'' = 2i\frac{\mu_2}{\mu_1 + \mu_2}}$$

$$i' = i - i'' = \frac{i\mu_1 + i\mu_2 - 2i\mu_2}{\mu_1 + \mu_2}$$

$$\boxed{i' = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\, i}$$

ⓑ in II

attractive if $\mu_1 > \mu_2$

repulsive if $\mu_2 > \mu_1$

(6)

Solution to Q.M. Problem

(a) $H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|\underset{\sim}{r_1} - \underset{\sim}{r_2}|)$

if $\underset{\sim}{r} = \underset{\sim}{r_1} - \underset{\sim}{r_2}$, $\mu = m_1 m_2/(m_1+m_2)$, $M = m_1+m_2$, $\vec{p} = (m_2\vec{p}_1 - m_1\vec{p}_2)/(m_1+m_2)$, $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$

$\underset{\sim}{P} = \underset{\sim}{p_1} + \underset{\sim}{p_2}$

$$H = \underbrace{\frac{p^2}{2\mu} + V(r)}_{H_{rel}} + \underbrace{P^2/2M}_{\text{Motion of center mass - uniform translation}}$$

(b) $p^2 = p_r^2 + \frac{\hat{L}_{op}^2(\theta\phi)}{r^2}$, where L^2 is the (angular momentum)^2 operator

$$H = \frac{P^2}{2M} + \frac{p_r^2}{2\mu} + V(r) + \frac{\hat{L}^2(\theta\phi)}{2\mu r^2}$$

so the variables separate

$$\Psi(\underset{\sim}{r_1}\underset{\sim}{r_2}) = \Phi(\underset{\sim}{R})\psi(r)Y(\theta\phi)$$

(1) $\frac{P^2}{2M}\phi(R) = E_{cm}\phi(R)$: C.M. motion

(2) $\left(\frac{p_r^2}{2\mu} + V(r)\right)\psi(r) = E_v\,\psi(r)$: Vibrational motion

(3) $\frac{\hat{L}^2}{2\mu r_0^2}Y(\theta\phi) = E_{rot}\,Y(\theta\phi)$: Rotational motion (with r_0 a fixed parameter)

(c) Eq (3) has the spherical harmonics as solution

$$\frac{L^2}{2\mu r^2}Y(\theta\phi) = \frac{l(l+1)\hbar^2}{2\mu r_0^2}Y(\theta\phi)$$

($\mu = M/2$) $\quad E_l = \frac{l(l+1)\hbar^2}{2\mu r_0^2} = \frac{l(l+1)\hbar^2}{M r_0^2} \quad l = 0, 1, 2 \ldots$

$\Delta E_{10} = E_1 - E_0 = 2\hbar^2/2\mu r_0^2 = 2\hbar^2/M r_0$ $\quad$ ($\mu = M/2$)

$$\Delta E = \frac{2\hbar^2}{m r_0} = \frac{2(\hbar c)^2}{mc^2 r_0^2}$$

(give
$\hbar =$
$c =$
$m_p =$
$k =$)

$\hbar c = 197 \text{ MeV Fm} = 197 \text{ MeV} \cdot 10^{-13} \text{ cm}$
$m_p c^2 = 938.3 \text{ MeV}$
$r_0 = .74 \text{ Å} = .74 \cdot 10^{-8} \text{ cm}$

$$\Delta E = \frac{2(197 \text{ MeV} \cdot 10^{-13} \text{ cm})^2}{(938 \text{ MeV})(.74 \cdot 10^{-8})^2} = \underline{.015 \text{ eV}}$$

d) kT at room temperature is $\sim 1/40$ eV $= .025$

so the lowest level is excited

(give
$k = 1\text{eV}/11604\,°K$)