PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #26

January 8, 1977

General Instructions

This Comprehensive Examination for Winter 1977 (#26) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

Some information you may find useful:

$$\nabla \times \phi\vec{A} = \phi\nabla\times\vec{A} + (\nabla\phi)\times\vec{A}$$

$$\nabla\cdot\phi\vec{A} = \phi\nabla\cdot\vec{A} + \nabla\phi\cdot\vec{A}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\frac{d\vec{A}}{ds} = \left(\frac{d\vec{r}}{ds}\cdot\nabla\right)\vec{A}$$

$$\nabla = \frac{\vec{r}}{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$(1+x)^n = \sum_{k=0}^{\infty}\frac{n!}{k!(n-k)!}x^k$$

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$2\pi\delta(q-q') = \int_{-\infty}^{\infty} e^{-i(q-q')s}\,ds$$

$$\ln n! \approx \frac{1}{2}\ln 2\pi n + n\ln n - n$$

$h = 6.6 \times 10^{-34}$ joule-sec

$c = 3 \times 10^{8}$ m/sec

$a_0 = 0.5 \times 10^{-10}$ m

$k = 1.38 \times 10^{-23}$ joule/°K

$(1-x)^{-1} = 1+x+x^2+x^3+\ldots.$

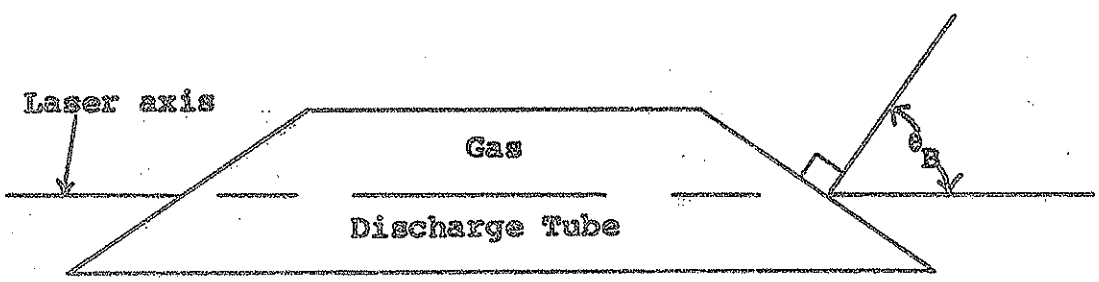
$(1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots.$

1. (a) For a typical He-Ne laser, operating in the TEM_{00} mode, laser oscillation occurs at each of a small group of adjacent closely spaced resonant frequencies. Each of these frequencies corresponds to a longitudinal, or axial, mode of the laser cavity. Derive the equation which determines the allowed axial mode frequencies.

(6)

(b) In many gas discharge lasers the planar end windows of the discharge tube are not mounted with their faces perpendicular to the optical axis of the laser; but, instead, they are mounted at the Brewster angle, θ_B, as indicated in the figure below. State, or derive,

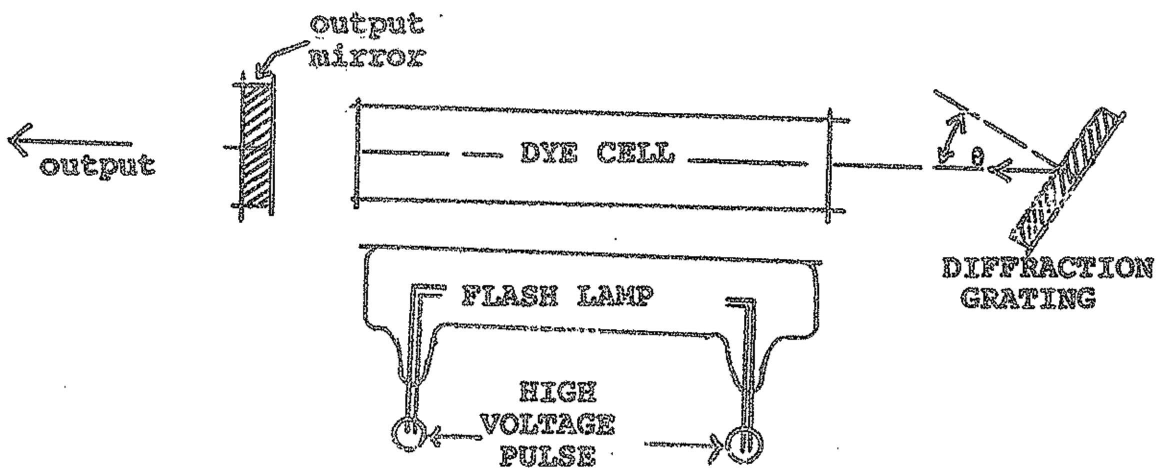
(7)



the equation relating θ_B to the index of refraction of the end window material, and state the purpose for mounting the end windows in this configuration.

1 (continued).

(7) (c) In many tunable dye lasers a totally reflecting diffraction grating (often called a reflection grating) is used as the tuning element, as indicated in the figure. Starting with "the grating equation" (if you can't remember the grating equation, derive it), derive the relation between the angle θ, the grating spacing d, and the laser output wavelength, λ.



Schematic of a dye laser, using flashlamp excitation and a diffraction grating for tuning.

2. (a) Consider a large number N of ideal monatomic gas molecules at temperature T, confined within a cubical box
(10) of length L, volume V. Derive, from a classical microscopic kinetic theory calculation, the ideal gas law describing the molecules in the box,

$$PV = NkT.$$

(b) Consider a large number N of quantum-mechanical
(10) oscillators at temperature T. The oscillators are all identical and have frequency ω.

(i) Show that the average value of the internal energy for these oscillators is

$$\bar{U} = \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}$$

(You may use a classical distribution law.)

(ii) Calculate the heat capacity for this system at constant volume and comment on the low ($T\to 0$) and high ($T\to\infty$) limits.

3\. (a) The total parity of a system is the product of the parity of its spin-space wave function and an additional number, known as the intrinsic parity. The intrinsic parity of the pi meson can be determined from the (parity-conserving
(5) strong interaction) reaction

$$\pi^- + d \rightarrow n + n.$$

The deuteron has an intrinsic spin of one unit and even parity, and the π^- has intrinsic spin of zero. The π^- is captured by the deuteron from a hydrogen-atom-type 1s level. Determine the intrinsic parity of the π^- from this information.

(b) The quark model postulates three quarks of charges $+\frac{2}{3}$, $-\frac{1}{3}$, and $-\frac{1}{3}$, with respective strangenesses 0, 0, and -1. All have spin $\frac{1}{2}$. The Ω^- particle is composed of 3 quarks and has spin $\frac{3}{2}$, charge -1, and strangeness -3. Discuss
(5) the internal composition of the Ω^- in terms of the quark model. In particular, what fundamental principle of quantum physics appears to be violated, and what modifications of the model appear necessary to overcome this violation?

(c) The nucleus W^{182} has a permanent, quadrupole-type deformation (and therefore has rotational energy levels). The ground state of this even-even nucleus has zero spin
(5) and even parity, and the first excited state is 100 keV above the ground state. Calculate the energy of the second excited state.

(d) The helicity of a particle is the projection of its intrinsic spin on the direction of its linear momentum.
(5) All neutrinos observed in nature have helicity $-\frac{1}{2}$; that is, their spins and linear momenta are always anti-parallel. How does this definite helicity require that the rest mass of the neutrino be identically zero?

4\. (a) Consider an electromagnetic wave propagating in a region of space free of charges and currents, and characterized by (non-constant) permittivity ε and permeability μ. Suppose the wave to be of the form

$$\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r})e^{-i\omega t} \qquad \vec{H}(\vec{r},t) = \vec{H}_0(\vec{r})e^{-i\omega t}$$

(12) with

$$\vec{E}_0(\vec{r}) = \vec{e}(\vec{r})e^{ikf(\vec{r})} \qquad \vec{H}_0(\vec{r}) = \vec{h}(\vec{r})e^{ikf(\vec{r})}$$

$\vec{e}$ and $\vec{h}$ are vector functions of position, k is the wave number $(= \frac{\omega}{c})$, and $f(\vec{r})$ is a scalar function of position. Use Maxwell's equations to show that, in the optical range where λ is small ($kr >> 1$ - neglect terms in $\frac{1}{k}$),

$$(\nabla f)^2 = n^2$$

where n is the index of refraction of the medium.
Note: The surfaces $f(\vec{r})$ = constant are the wave fronts, and the rays are normal to these surfaces.

(b) Consider a spherically symmetric medium such as the earth's atmosphere, where $n = n(r)$. Use the result of part (a) to show that for such a medium, the rays of light propagate according to $\vec{r} \times n\hat{u}$ = constant

(8) along the ray. In this expression,
$\vec{r}$ locates a point on the ray relative to the origin of coordinates, and $\hat{u}$ is a unit vector along the ray (i.e., normal to $f(\vec{r})$ = constant).
Hint: Let s be a coordinate along the ray. Show that $\frac{d}{ds}(\vec{r} \times n\hat{u}) = 0$.

5. A particle of mass m moves in a potential

$$V(x) = mgx.$$

(10) (a) Solve the Heisenberg equations of motion for $\langle x\rangle$ and $\langle p\rangle$ as functions of time. Comment on your results. Hint: Evaluate $\frac{d}{dt}\langle x\rangle$ and $\frac{d}{dt}\langle p\rangle$.

(10) (b) Find the allowed energies for this potential by solving Schroedinger's Equation in momentum space; i.e.

$$\hat{H}(p,x)\phi_E(p) = E\phi_E(p).$$

(i) What values of E are allowed?

(ii) Are the energy levels degenerate?

(iii) Find the energy eigenfunctions $\phi_E(p)$ normalized such that

$$\int \phi_E^*(p)\phi_{E'}(p)dp = \delta(E-E').$$

6. Consider a system of N non-interacting classical particles (N is of the order of Avogadro's number) in which the energy of each particle can assume only two distinct values, 0 and E (E>0). Denote by n_0 and n_1 the occupation numbers of the energy level 0 and E , respectively.

(7) (a) Determine (in terms of N, n_0, and n_1 only) the number of ways of attaining a given distribution of the particles, and then determine the most likely distribution. (No credit will be given for guesses.)

(7) (b) Suppose the system to be in its state of maximum probability, and determine its entropy.

(6) (c) Estimate the likelihood of fluctuations between the system states of maximum and minimum probability. Assume the availability of any necessary heat reservoirs.

(1) ~~OTPG~~ OPTICS ANSWERS

a) For ~~r~~esonance $n\lambda = 2L$, or $\lambda_n = \frac{2L}{n}$.

But $\lambda_n = \frac{c}{\nu_n}$; $\therefore \frac{c}{\nu_n} = \frac{2L}{n}$, or $\nu_n = \frac{nc}{2L}$.

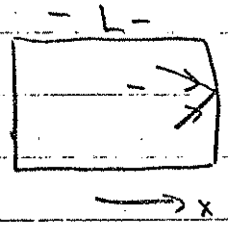
b) $\tan\theta_B = n$. At θ_B, the light loss due to ~~ref~~ window reflection is zero for light polarized in the plane of incidence; $\therefore$, for this polarization, the threshold for gain is substantially lower than for the other polarization. The laser output light will always be linearly polarized, with polarization in the plane of incidence.

c) $n\lambda = d(\sin i + \sin r)$. Here, $i = r = \theta$; $\therefore$ for first order operation at λ, $\lambda = 2d\sin\theta$.

(2)

Undergraduate Thermo

(A) consider 1 collision



$$F_x = \frac{\Delta P_x}{\Delta t} = \frac{2mV_x}{\Delta t}$$! Force on wall

$$\Delta t = 2L/V_x$$

$$F_x = \frac{2mV_x}{2L/V_x} = \frac{m\overline{V_x}^2}{L}$$

so for many collisions; $\overline{F_x} = \frac{M}{L}\overline{V_x^2} = \frac{M}{L}\frac{1}{3}\overline{V^2}$

, N molecules:

pressure on wall $= N\overline{F}/L^2 = P = N\frac{M}{L^3}\frac{1}{3}\overline{V}^2 = N\frac{1}{3}M\overline{V}^2/V$

$$PV = \frac{1}{3}M\overline{V}^2_N = \frac{1}{3}\,2\times\overline{KE} = \frac{2}{3}N\,\overline{KE}$$

$$\overline{KE} = \frac{3}{2}kT \qquad (\tfrac{1}{2}kT / \text{degree freedom})$$

$$PV = \cancel{\tfrac{2}{3}}\cancel{\tfrac{3}{2}}NkT$$

$$\boxed{PV = NkT}$$

Undergraduate Thermo

① (A) Consider a large number N of ideal, monatomic gas molecules, at temperature T confined within a cubical box of length L, volume V.

Derive, from a classical microscopic, kinetic theory calculation, the ideal gas law describing the molecules in the box,

$$PV = NkT.$$

(B) Consider a large number, N of quantum mechanical oscillators ~~each~~ at temperature T. The oscillators are all identical and have frequency ω.

(i) Calculate the average value of internal energy for these oscillators (you may use classical distribution laws).

(ii) Calculate the heat capacity for this system at constant volume and comment on the low ($T \to 0$) and high ($T \to \infty$) limits.

(3) Undergraduate Q. 4
Solution

(a) total a.m of $\pi^- + d$ = spin of π
+ spin of d
+ orbital a.m of $(\pi - d)$
$= 0 + 1 + 0 = 1$

$\Rightarrow$ total a.m of $n + n = 1$

possible configuration of $n+n$: $\uparrow\uparrow$ $(l = 0$ or $l = 2)$
$\downarrow\uparrow$ $(l = 1)$

First is not possible if total w.f. is to be antisymm.
$\therefore$ $l = 1$ for $n + n$ $\Rightarrow$ $\pi = (-1)^l = -1$
$\therefore$ parity of $\pi^- + d = -1$
parity of $d = +1$
parity of orbital motion $= (-1)^l = +1$
$\Rightarrow$ π^- has odd parity.

(b) 3 quarks with $j = \frac{1}{2}$, $q = -\frac{1}{3}$, $S = -1$ with spin aligned - violates Pauli principle unless there is an additional quantum number which distinguishes the 3 quarks. $\rightarrow$ color

(c) $E \propto J(J+1)$

Spectrum $0^+, 2^+, 4^+$ $\qquad \dfrac{E_{4^+}}{E_{2^+}} = \dfrac{4\cdot 5}{2\cdot 3} = \dfrac{10}{3}$

$\Rightarrow E_{4^+} = 330$ keV

(d) definite helicity $\Rightarrow$ same for all reference frames $\Rightarrow$ can't transform to frame of reference in which $\vec{p}$ changes sign $\Rightarrow$ neutrino moves at speed of light $\Rightarrow$ $m = 0$.

(4) Graduate Optics - Solution

(a) $\nabla\times\vec{H} = \frac{\partial D}{\partial t} \longrightarrow \nabla\times\vec{H}_0 e^{-i\omega t} = \epsilon E_0(-i\omega)e^{-i\omega t}$

$$\boxed{\nabla\times\vec{H}_0 + i\omega\epsilon\vec{E}_0 = 0} \quad (1)$$

$\nabla\times\vec{E} + \frac{\partial b}{\partial t} = 0 \longrightarrow \nabla\times E_0 e^{-i\omega t} + \mu\vec{H}_0(-i\omega)e^{-i\omega t} = 0$

$$\boxed{\nabla\times\vec{E}_0 - i\omega\mu\vec{H}_0 = 0} \quad (2)$$

$$\boxed{\nabla\cdot\epsilon E_0 = 0} \quad (3)$$

$$\boxed{\nabla\cdot\mu\vec{H}_0 = 0} \quad (4)$$

$\nabla\times\vec{H}_0 = \nabla\times\left(\hat{h}\,e^{ikf(\vec{r})}\right) = e^{ikf}\,\nabla\times\hat{h} + \left(\nabla e^{ikf}\right)\times\hat{h}$

[Given: $\nabla\times\phi\vec{A} = \phi\,\nabla\times\vec{A} + (\nabla\phi)\times\vec{A}$]

$= e^{ikf}\,\nabla\times\vec{h} + ik\,e^{ikf}\,\nabla f\times\vec{h} = e^{ikf}\left(\nabla\times\hat{h} + ik\,\nabla f\times\hat{h}\right)$

(1) $\Rightarrow \quad e^{ikf}\left(\nabla\times\hat{h} + ik\nabla f\times\hat{h}\right) + i\omega\epsilon\vec{e}\,e^{ikf} = 0$

$\nabla\times\vec{h} + ik\,\nabla f\times\vec{h} + i(ck)\,\epsilon\,\vec{e} = 0$

limit of large k $\quad \nabla f\times\vec{h} + c\epsilon\vec{e} = 0$

Similarly (2) $\Rightarrow \quad \nabla f\times\vec{e} - c\mu\vec{h} = 0$

(3) $\Rightarrow \quad \nabla\cdot\epsilon\vec{E}_0 = \nabla\cdot\epsilon\vec{e}\,e^{ikf} = \nabla\cdot\left(\vec{e}\,\epsilon e^{ikf}\right)$

[Given: $\nabla\cdot\phi\vec{A} = \phi\,\nabla\cdot\vec{A} + (\nabla\phi)\cdot\vec{A}$]

$= \left(\nabla\epsilon e^{ikf}\right)\cdot\vec{e} + \epsilon e^{ikf}\,\nabla\cdot\vec{e}$

$= \left(\epsilon\nabla e^{ikf} + e^{ikf}\nabla\epsilon\right)\cdot\vec{e} + \epsilon e^{ikf}\,\nabla\cdot\vec{e} = 0$

$= \left(ik\epsilon e^{ikf}\nabla f + e^{ikf}\nabla\epsilon\right)\cdot\vec{e} + \epsilon e^{ikf}\,\nabla\cdot\vec{e} = 0$

large k $\quad \nabla f\cdot\vec{e} = 0$

Similarly (4) $\Rightarrow \quad \nabla f\cdot\vec{h} = 0$

combine ① & ②

$$① \Rightarrow \qquad \vec{e} = -\frac{1}{c\epsilon}\left(\nabla f \times \vec{h}\right)$$

plug into ② $\qquad \nabla f \times \left(-\frac{1}{c\epsilon} \nabla f \times \vec{h}\right) - c\mu \vec{h} = 0$

$$-\frac{1}{c\epsilon}\left(\nabla f \times (\nabla f \times \vec{h})\right) - c\mu \vec{h} = 0$$

Given: $\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$

$$= -\frac{1}{c\epsilon}\left[\nabla f \, \cancel{(\nabla f \cdot \vec{h})}^{0} - \vec{h}\,(\nabla f)^2\right] - c\mu \vec{h} = 0$$

$$(\nabla f)^2 = c^2 \epsilon\mu = c^2/v^2 = n^2$$

$$\boxed{(\nabla f)^2 = n^2}$$

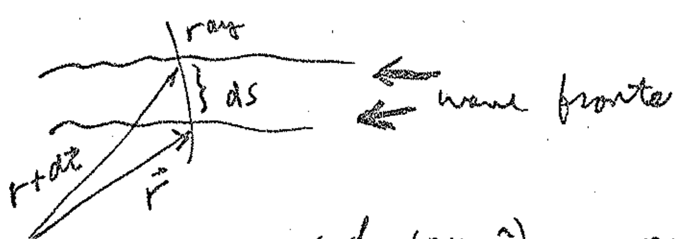
(b) $\qquad |\nabla f| = |n|$

∇f is perpendicular to $f =$ constant $\Rightarrow \hat{u} = \frac{\nabla f}{|\nabla f|} = \frac{\nabla f}{n}$

let s be a coordinate measured along the ray. We want to show that $\qquad \frac{d}{ds}(\vec{r} \times n\hat{u}) = 0$

$$\frac{d}{ds}(\vec{r} \times n\hat{u}) = \frac{d\vec{r}}{ds} \times n\hat{u} + \vec{r} \times \frac{d}{ds} n\hat{u}$$

$$|d\vec{r}| = |ds| \Rightarrow \frac{d\vec{r}}{ds} = \hat{u} \Rightarrow \frac{d\vec{r}}{ds} \times \hat{u} = 0$$



$$\therefore \frac{d}{ds}(r \times n\hat{u}) = r \times \frac{d}{ds}(n\hat{u}) = \vec{r} \times \frac{d}{ds}(\nabla f)$$

$$= \cancel{\vec{r} \times \frac{d}{ds}} = \vec{r} \times \left(\frac{d\vec{r}}{ds} \cdot \nabla\right) \nabla f$$

[Given: $\frac{d\vec{A}}{ds} = \left(\frac{d\vec{r}}{ds} \cdot \nabla\right)\vec{A}$]

$$= \hat{r} \times \left(\frac{\nabla f}{n} \cdot \nabla\right) \nabla f = \frac{1}{n}\, \hat{r} \times \left(\frac{1}{2} \nabla (\nabla f \cdot \nabla f)\right)$$

$$= \frac{1}{2n}\, \hat{r} \times \nabla n^2$$

$$= \frac{1}{2n}\, \vec{r} \times 2n \nabla n = \hat{r} \times \nabla n$$

$$\left[\text{Given } \nabla = \frac{\vec{r}}{r}\frac{\partial}{\partial r} + \cdots\right]$$

$$= \vec{r} \times \left(\frac{\vec{r}}{r}\frac{\partial n}{\partial r}\right) = 0$$

Q.E.D.

~~(c) $\vec{r} \times m\vec{v}$ = constant (angular momentum) under the influence of a central force $F = F(r)$~~

$n\hat{u} = \nabla f$

$\vec{r} \times \nabla f$

(5)

solution to Grad Q.M.

$V(x) = mgx$

(a) $\langle p \rangle, \langle x \rangle$

$$\hat{H} = p^2/2m + V = p^2/2m + mgx$$

$$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\langle [x, \hat{H}] \rangle + \frac{\partial x}{\partial t}$$

$\rightarrow$ assume no explicit time dependence (eg constants)

$$\frac{d\langle p\rangle}{dt} = \frac{1}{i\hbar}\langle [p, \hat{H}] \rangle$$

$$[x, \hat{H}] = [x, \tfrac{p^2}{2m} + mgx] = \tfrac{1}{2m}[x, p^2] + m\cancel{[x, x]}^{\rightarrow 0}$$

$$= \tfrac{1}{2m}\{2p[x,p]\} = \tfrac{1}{2m} 2p\, i\hbar = \hbar i p/m$$

$$[p, H] = [p, p^2/2m + mgx] = \underbrace{[p, p^2/2m]}_{\rightarrow 0} + mg[p, x] = -mg\hbar i$$

$$\frac{d}{dt}\langle x \rangle = \frac{1}{i\hbar}\langle \hbar i p\rangle = \langle p\rangle/m$$

$$\frac{d}{dt}\langle p \rangle = \frac{1}{i\hbar}(-mg\hbar i) = -mg = \underbrace{\langle F(x)\rangle = \langle -\tfrac{\partial V}{\partial x}\rangle}$$

Classical Force

$$(\neq F(\langle x\rangle))$$

These are the Ehrenfest Equations

only get this classical result if very small wavepacket w.r.t. potential

$$\frac{d^2}{dt^2}\langle x\rangle = \frac{1}{m}\frac{d\langle p\rangle}{dt} = -\frac{mg}{m} = -g$$

$$\frac{d\langle p\rangle}{dt} = -mg \implies \boxed{\langle p\rangle = -mgt + \langle p_0\rangle}$$

$$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m} = \frac{-mgt + \langle p_0\rangle}{m}$$

$$\implies \boxed{\langle x\rangle = \frac{-gt^2}{2} + \frac{\langle p_0\rangle t}{m} + \langle x_0\rangle}$$

These are the same results as obtained classically with a

Momentum Space

$$\hat{H}\,\phi_E(p) = E\,\phi_E(p)$$

$$\hat{H} = p^2/2m + mgx = p^2/2m + mg\hbar i\frac{\partial}{\partial p}$$

$$\frac{p^2}{2m}\phi(p) + mg\hbar i\frac{\partial\phi(p)}{\partial p} - E\phi(p) = 0$$

$$\partial\phi/\partial p = -i\left[\frac{E}{mg\hbar} - \frac{p^2}{2m^2g\hbar}\right]\phi(p) = \frac{-i}{mg\hbar}(E - p^2/2m)\phi(p)$$

$$\phi_E(p) = \underset{\uparrow\ \text{Constant}}{A_E}\, e^{-\frac{i}{mg\hbar}[Ep - p^3/6m]} = A e^{-i\frac{Ep}{mg\hbar}} e^{+i\,p^3/6m^2g\hbar}$$

① all values of E allowed
② NO, (only first order derivatives occur) ouesalt/≡
③ $\int \phi^*_E(p)\,\phi_{E'}(p)\,dp = \delta(E-E')$

$$\delta(E-E') = A^*_E\,A_E\int_{-\infty}^{\infty} dp\, e^{ip(E-E')/mg\hbar}$$

$$\left(\int_{-\infty}^{\infty} e^{-i(E-E')x}\,dx = 2\pi\,\delta(E-E')\right)$$

let $x = p/mg\hbar$

$$\delta(E-E') = A^*_E\,A_E\,mg\hbar\,\delta(E-E')\,2\pi$$

let $A = 1/\sqrt{mg\hbar 2\pi}$

$$\boxed{\phi_E(p) = \frac{1}{\sqrt{2\pi mg\hbar}}\, e^{-iEp/mg\hbar}\, e^{i p^3/6m^2g\hbar}}$$

$$\int \phi_E(p)\, \phi_E^*(p')\, dE = \delta(p-p')$$

$$= \left(\frac{1}{Mg\hbar(2\pi)} \int_{-\infty}^{\infty} e^{\frac{-iE}{Mg\hbar}[p-p']} \, dE\right) \times e^{i(p^3-p'^3)/6m^2g\hbar}$$

$$= \left(\frac{1}{Mg\hbar(2\pi)}\, mg\hbar \;\; 2\pi\, \delta(p-p')\right) e^{i(p^3-p'^3)/6m^2g\hbar}$$

$$= \delta(p-p') \checkmark$$

6 Stat Mech Answers

a) $W = \dfrac{N!}{n_0!\, n_1!} = \dfrac{N!}{(N-N_1)!\, N_1!}$. To determine

W_{max} use $\ln n! \simeq n \ln n$ and set $\dfrac{\partial(\ln W)}{\partial N_1} = 0$

$$\ln W \simeq N \ln N - (N-N_1)\ln(N-N_1) - N_1 \ln N_1$$

$$\frac{\partial(\ln W)}{\partial N_1} \simeq +\ln(N-N_1) + \frac{N-N_1}{N-N_1} - \ln N_1 - \frac{N_1}{N_1} = 0$$

$$\therefore \ln\left(\frac{N-N_1^{max}}{N_1^{max}}\right) = 0, \text{ or } N - N_1^{max} = N_1^{max} \Rightarrow N_1^{max} = \frac{N}{2}$$

b) $S = \cancel{k} \ln W_{max} = k \ln\left[\dfrac{N!}{\left(\frac{N}{2}!\right)^2}\right]$

$$\frac{S}{k} = \ln(N!) - 2\ln \frac{N}{2}! \simeq N \ln N - N \ln \frac{N}{2} \simeq N \ln 2$$

$$S = k \cdot N \ln 2$$

c) $W_{min} = \dfrac{N!}{0!\, N!} = 1$; $\therefore \dfrac{W_{max}}{W_{min}} = W_{max} = \dfrac{N!}{\left(\frac{N}{2}!\right)^2}$

$$\ln W_{max} \simeq N \ln N - N \ln N/2 \simeq N; \therefore W_{max} \simeq e^N.$$