# PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #25

October 2, 1976

## General Instructions

This Comprehensive Examination for Fall 1976 (#25) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil and bluebook, on the floor.

Some information you may find useful:

$$\int \sin^2 x\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$$

$$\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \quad \text{for } m^2 \neq n^2$$

$$\int_0^\infty x^{2a} e^{-r^2x^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2a-1)\sqrt{\pi}}{2^{a+1} r^{2a+1}}$$

$$(1+X)^n = \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} X^k$$

$h = 6.6 \times 10^{-34}$ joule-sec

$c = 3 \times 10^{8}$ m/sec

$a_0 = 0.5 \times 10^{-10}$ m

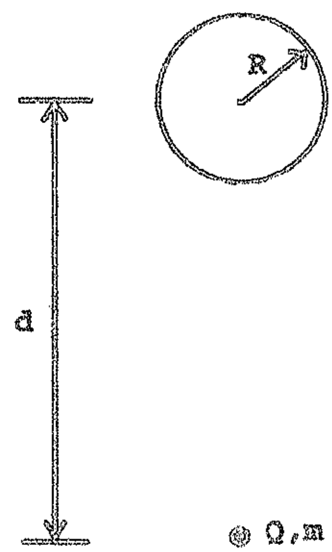
$k = 1.38 \times 10^{-23}$ joule/°K

Answer all questions. Each question is worth 20 points.
Show all work so that part credit may be given where justified.

(10) 1. (a) A 400 keV positron annihilates with an electron which is at rest. Determine, approximately, the maximum and minimum energies of the two gamma rays produced as a result of the annihilation.

(10) (b) Derive the Einstein velocity addition law for parallel velocities.

(20) 2. A point charge Q of mass m is to be placed at rest at its equilibrium position a distance d directly below the center of an uncharged conducting sphere of radius $R \ll d$. Find d and the frequency of small oscillations in the vertical direction.



3\. Given N pairs of particles in a very large box of volume V. The particles in each pair are connected to each other by a spring. When the pairs lie within the box their Hamiltonian is

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + \frac{s}{2}(\vec{r}_1 - \vec{r}_2)^2$$

The N pairs of particles are to be regarded as an ideal gas in thermal equilibrium at a temperature T.

(10) (a) Show that the partition function for this system is proportional to Z, where

$$Z = (m_1 kT)^{3/2} \ (m_2 kT)^{3/2} \ \left(\frac{kT}{s}\right)^{3/2} V$$

Calculate from Z:

(b) $C_V$, and compare it to the well-known result for a gas of diatomic molecules.

(2) (c) $\frac{1}{2} s \langle (\vec{r}_1 - \vec{r}_2)^2 \rangle$

(2) (d) $\langle (\vec{p}_1)^2 \ (x_1 - x_2)^{24} \rangle \ - \ \langle \vec{p}_1^{\,2} \rangle \langle (x_1 - x_2)^{24} \rangle$

<u>Hint:</u> Parts b, c, and d can be done without any explicit integration, although one should not just guess the answers.

4. An electromagnetic wave is traveling through empty space. Its electric field is of the form

$$\vec{E} = E_0 \cos k_1 x \cos (k_2 z - \omega t)\, \widehat{e_y}$$

($\widehat{e_y}$ is a unit vector in the y direction).

(2) (a) What are the magnitude and direction of the phase velocity?

(8) (b) Determine the associated magnetic field $\vec{B}$.

(6) (c) Determine $k_2$ as a function of $\omega$ and $k_1$.

(2) (d) Where can perfectly conducting walls be placed which will confine this wave, yet otherwise not change the form of $\vec{E}$?

(2) (e) At what rate per unit area is power transmitted in the direction of the wave's motion at

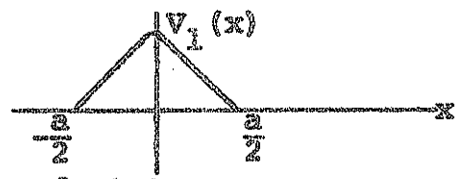
$$x = y = z = t = 0?$$

5. (a) A particle of mass m is constrained to move freely in one
(10) dimension in the region $x = -\frac{a}{2}$ to $x = \frac{a}{2}$. For each of the weak perturbations $V_1(x)$ described below, tell whether the energy of the particle is increased, decreased, or unchanged, and briefly justify your answer. Unless otherwise instructed, consider first-order perturbations only, and take $b>0$.

(i) $V_1(x) = bx$, $-\frac{a}{2}\leq x\leq\frac{a}{2}$. The particle is in its ground state (corresponding to the lowest-lying non-trivial wave function).

(ii) $V_1(x) = bx$, $-\frac{a}{2}\leq x\leq\frac{a}{2}$. The particle is in the first excited state.

(iii) $V_1(x) = b(x+\frac{a}{2})$, $-\frac{a}{2}\leq x\leq 0$;
$= b(\frac{a}{2}-x)$, $0\leq x\leq +\frac{a}{2}$




The particle is in the ground state.

(iv) Same as (iii). The particle is in the first excited state.

(v) $V_1(x) = bx$, $-\frac{a}{2}\leq x\leq\frac{a}{2}$. The particle is in the ground state. Consider second-order perturbations.

(10) (b) A particle of mass m is constrained to move freely in one dimension in the region $x = 0$ to $x = a$. A weak gravitational field of strength g is in the +x direction. (Take the zero of gravitational potential at the origin.) Find the energy of the particle to first order in g and discuss the result. Consider in particular the applicability of the correspondence principle.

6. A gas of hydrogen atoms, all populating the ground 1s $^2S_{1/2}$ levels, can be produced by heating $H_2$ gas in an oven. Suppose such a gas of H atoms is bombarded with a short burst of unpolarized and isotropically directed photons. The wavelengths of these photons are confined to a narrow band centered at 121.6 nm (10.2 eV). At the end of the photon pulse, which may be regarded as a $\delta$-function in time, some of the H atoms will populate the 2p $^2P_{3/2,1/2}$ levels; and their ensuing radiative decay will be exponential, as illustrated in Fig. 1. If the experiment is now repeated, but with a small uniform electric field applied to the region of excitation, the radiative decay exhibits the oscillatory structure shown in Fig. 2.

(5) (a) Explain qualitatively the cause for the difference between Figs. 1 and 2.

(10) (b) Show how the data of Figs. 1 and 2 are related to time-dependent quantum mechanical descriptions of excited H atoms.

(5) (c) From the information given, estimate the energy (in eV) of the Lamb shift in hydrogen.

<u>Hint:</u> The relevant H atom energy levels are displayed schematically in Fig. 3.

<u>Figures are on page 7.</u>

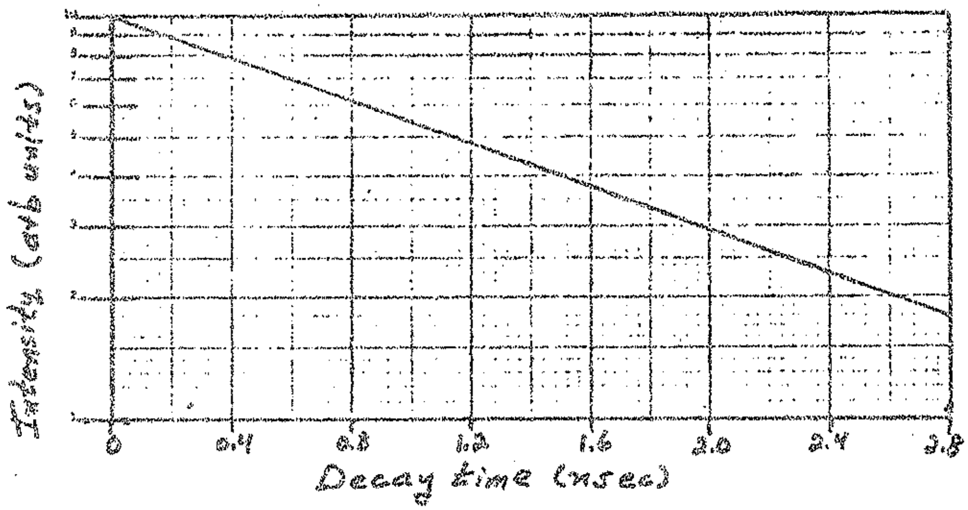



Fig. 1

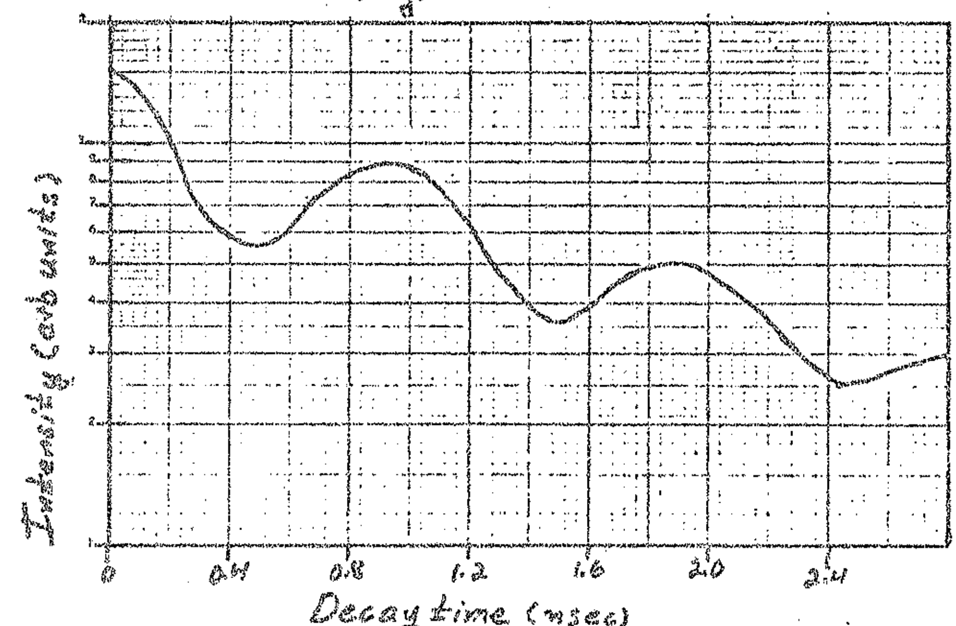



Fig. 2

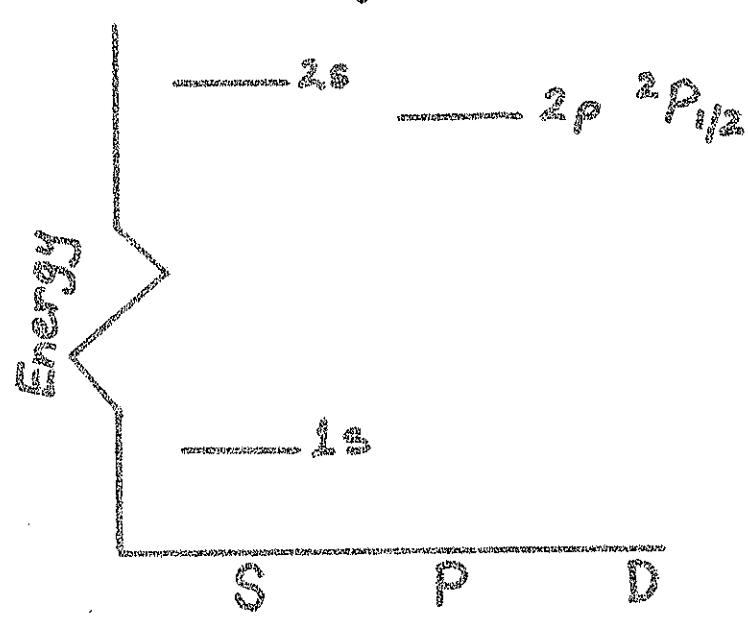



Fig. 3

#1

Ans: a) The max & min energy $\gamma$-rays travel $\parallel$ and anti-parallel, respectively to the direction of the incident positron; $\therefore$, we have

$$mc^2(\gamma) = E_{\gamma_1} + E_{\gamma_2}, \text{ and}$$

$$mv\gamma = (E_{\gamma_1} - E_{\gamma_2})/c, \text{ which give}$$

$$mc^2(\gamma + \beta\gamma) = 2E_{\gamma_1} \text{ and } mc^2(\gamma - \beta\gamma) = 2E_{\gamma_2}$$

To determine $\gamma$ use $E_i = mc^2(\gamma - 1)$, $\gamma = 1 + \frac{E_i}{mc^2}$)

or $\gamma \simeq 1.8$. For $\beta$, use $\frac{1}{\gamma^2} = 1 - \beta^2$, or

$$\beta^2 = \frac{\gamma^2 - 1}{\gamma^2}; \; \therefore \beta\gamma = \sqrt{\gamma^2 - 1} \simeq 1.5. \text{ Thus}$$

$$E_{\gamma_1} \simeq \frac{mc^2}{2}(3.3) \;\&\; E_{\gamma_2} \simeq \frac{mc^2}{2}(0.3)$$

$\boxed{E_1 = 828 \text{ keV} \quad E_2 = 72 \text{ keV}}$

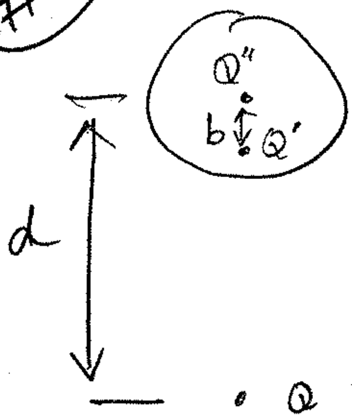
check: $E_{\gamma_1} + E_{\gamma_2} \simeq \frac{mc^2}{2} \cdot 3.6$ .OK.

b)
$$\begin{pmatrix} 0 \\ 0 \\ \gamma''\beta'' \\ i\gamma'' \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \gamma & -i\beta\gamma \\ 0 & & i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \gamma'\beta' \\ i\gamma' \end{pmatrix}$$

$\gamma''\beta'' = \gamma(\gamma'\beta' + \beta\gamma')$; $i\gamma'' = i\gamma(\beta\beta'\gamma' + \gamma')$,
$= \gamma\gamma'(\beta' + \beta)$ $\quad \gamma'' = \gamma\gamma'(1 + \beta\beta')$, which

together give $\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}$

#2




For the sphere to be an equipotential

$$\frac{kQ}{d-R}+\frac{kQ'}{R-b}=\frac{kQ}{d+R}+\frac{kQ'}{b+R}$$

Take $V=0$ on surface due to $Q + Q'$;
add $Q''$ to change potential of surface.

$$\therefore \quad b=\frac{R^2}{d} \qquad Q'=-\frac{R}{d}Q$$

For sphere to be uncharged $Q''=-Q'$

Force on $Q$

$$F=\frac{kQQ'}{(d-b)^2}+\frac{kQQ''}{d^2}-mg$$

$$=kQ^2\frac{R}{d}\left[\frac{1}{(d-b)^2}-\frac{1}{d^2}\right]-mg$$

$$=kQ^2\frac{R}{d^3}\left[\frac{2}{d}\frac{R^2}{d}\right]-mg=0$$

$$\therefore \boxed{d=\sqrt[5]{2kq^2R^3/mg}}$$

let $d \to d + x$

$$F = \frac{2kQ^2 R^3}{(d+x)^5} - mg = m\ddot{x}$$

$$\frac{2kQ^2 R^3}{d^5}\left(1 - \frac{5x}{d}\right) - mg = m\ddot{x}$$

$$m\ddot{x} = -\frac{10kQ^2 R^3}{d^6}\, x$$

$$\boxed{\therefore \omega = \sqrt{\frac{10kQ^2R^3}{md^6}}}$$

## Solution ③ (Graduate Stat. Mech).

a) $\mathcal{H} = p_1^2/2m_1 + p_2^2/2m_2 + \frac{s}{2}(\vec{r}_1 - \vec{r}_2)^2$ (For 1 pair)

(10pts) $Z = \int e^{-\beta\mathcal{H}} d^3r_1 d^3r_2\, d^3p_1 d^3p_2$ $(\beta = 1/kT)$

$$= \int_0^\infty 4\pi p_1^2 dp_1 e^{-\beta p_1^2/2m_1} \int_0^\infty 4\pi p_2^2 e^{-\beta p_2^2/2m_2} \int d^3r_1 d^3r_2\, e^{-\beta\frac{s}{2}(r_1-r_2)^2}$$

let $r = r_1 - r_2$, $x = \frac{r_1+r_2}{2}$

$$\left(\frac{4\pi\sqrt{\pi}\,(2m_1)^{3/2}}{4\,\beta^{3/2}}\right) \left(\frac{4\pi\sqrt{\pi}\,(2m_1)^{3/2}}{4\,\beta^{3/2}}\right) \underbrace{\int d^3x}_{V} \int d^3r\, e^{-\frac{\beta s}{2}r^2} \quad V\left(\frac{4\pi\sqrt{\pi}\;2^{3/2}}{4\beta^{3/2}s^{3/2}}\right)$$

give

$$\boxed{\int_0^\infty x^{2a} e^{-r^2x^2} dx = \frac{1\cdot 3\cdot 5\cdots(2a-1)\sqrt{\pi}}{2^{a+1}\, r^{2a+1}}}$$

$$\underset{\text{(one pair)}}{Z_1} = \left(\frac{2\pi m_1}{\beta}\right)^{3/2}\left(\frac{2\pi m_2}{\beta}\right)^{3/2} V \left(\frac{2\pi}{\beta s}\right)^{3/2}, \quad Z_{all} \propto (Z_1)^N$$

$$\left(\ln Z_1 = \frac{3}{2}\ln\frac{m_1}{\beta} + \frac{3}{2}\ln\frac{m_2}{\beta} + \frac{3}{2}\ln\frac{1}{\beta s} + \ln V\right)$$

(b) (6pts) $\langle\mathcal{H}\rangle = -\frac{\partial}{\partial\beta}\ln Z\Big|_{N,V}$

$\ln Z = -\frac{9}{2}\ln\beta\, N + \text{const}$

(3pts) $\langle\mathcal{H}\rangle = \frac{9}{2}\,1/\beta\, N = \underline{\underline{\frac{9}{2}kT\,N}}$

(2pts) $C_v = \frac{\partial\langle\mathcal{H}\rangle}{\partial T}\Big|_{N,V} = \frac{\partial E}{\partial T}\Big|_{N,V} = \frac{9}{2}kN$

(2pts) At first this $\mathcal{H}$ looks like the one for a diatomic molecule, yet there we expect $C_V = 7/2\,k$. The difference is due to the molecule having a "spring" with nonzero unstretched length and ∴ only 1 degree of vibrational freedom. For a zero length spring, there are 3 degrees of freedom i.e. 2 more $\frac{1}{2}k$'s.

(c) (2 pts) $\frac{\partial Z_1}{\partial s} = \int d^3r_1\, r_2\, p_1\, p_2 \left[ -\frac{\beta}{2} (r_1 - r_2)^2 \right] e^{-\beta \mathcal{H}} = -\frac{\beta}{2} \langle (r_1 - r_2)^2 \rangle Z$

so $-\frac{\beta}{2} \langle (r_1 - r_2)^2 \rangle = \frac{1}{Z} \frac{\partial Z}{\partial s} = \frac{\partial \ln Z}{\partial s} = -\frac{3}{2s} N$

$\frac{1}{2} s \langle (\vec{r}_1 - \vec{r}_2)^2 \rangle = \frac{3}{2} kT N$

(d) (2 pts) Since the momentum and space integrals seperate, The two overages ae statistically independent, thus

$\langle (\vec{p}_1)^2 (x_1 - x_2)^{24} \rangle = \langle p_1^2 \rangle \langle (x_1 - x_2)^{24} \rangle$ So

answer $= 0$

## 4 Solutions / E.M.

$$\vec{E} = E_0 \cos k_1 x \cos(k_z z - \omega t)\,\hat{e}_y$$

(a) Phase velocity

constant phase $\Rightarrow$ $k_z z - \omega t = \text{const.}$,

$V = \omega/k_z$, in $+z$ direction

(b) $\vec{B} = ?$

let $\vec{B} = \mathrm{Re}\,(\vec{B}(r)\, e^{-i\omega t})$

$\vec{E} = \mathrm{Re}\,(E_0\, e^{i(k_z z - \omega t)} \cos k_1 x)\,\hat{e}_y$ (Done other way after (e) as check)

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\therefore \frac{i\omega}{c} B = \nabla \times E$$

$$= -\hat{e}_x\left(\cancel{\frac{\partial E_z}{\partial y}}^{\,0} - \frac{\partial E_y}{\partial z}\right) + \hat{e}_y\left(\cancel{\frac{\partial E_x}{\partial z}}^{\,0} - \cancel{\frac{\partial E_z}{\partial x}}^{\,0}\right) + \hat{e}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\frac{i\omega}{c} B = -\hat{e}_x E_0 \cos k_1 x\,(i k_z)\, e^{i(k_z z - \omega t)} + \hat{e}_z E_0(-k_1 \sin k_1 x)\, e^{i(k_z z - \omega t)}$$

$$B = -\frac{c E_0}{\omega} k_z \cos k_1 x\, e^{i(k_z z - \omega t)}\,\hat{e}_x + \frac{i c E_0}{\omega} k_1 \sin k_1 x\, e^{i(k_z z - \omega t)}\,\hat{e}_z$$

take real part

$$\vec{B} = -\frac{c E_0 k_z}{\omega} \cos k_1 x \cos(k_z z - \omega t)\,\hat{e}_x - \frac{c E_0 k_1}{\omega} \sin k_1 x \sin(k_z z - \omega t)\,\hat{e}_z$$

(c) $\nabla \times B = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \hat{e}_y\left[\frac{\omega E_0}{c} \cos k_1 x \sin(k_z z - \omega t)\right]$$

$$\nabla \times B\Big|_y = \hat{e}_y\left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right]$$

$$= \hat{e}_y\left[+\frac{c E_0 k_z^2}{\omega} \cos k_1 x \sin(k_z z - \omega t) + \frac{c E_0 k_1^2}{\omega} \cos k_1 x \sin(\;)\right]$$

compare these two : $c\frac{k_z^2}{\omega} + \frac{c k_1^2}{\omega} = \frac{\omega}{c}$

$$k_z^2 = \frac{\omega^2}{c^2} - k_1^2$$

Soltns

(d) Wherever $\underset{\sim}{E}$ has a node for all times, i.e. $\left\{ \begin{array}{l} k_1 x = n \cdot \frac{\pi}{2} \\ x = \frac{\pi}{2} \frac{n}{k_1} \end{array} \right.$

with n odd integer ; yet also need

$$k_2^2 = \frac{\omega^2}{c^2} - k_1^2$$

$$\text{with } k_2^2 > 0$$

$$k_1 < \frac{\omega}{c}$$

(e) Flux

at $x = y = z = t = 0$

$$\vec{E} = E_0 \hat{e}_y$$

$$\vec{B} = + \frac{k_2 E_0 c}{\omega} \hat{e}_x$$

$$\underset{\sim}{S} = \text{Power/area} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$= \frac{c^2}{4\pi} \frac{k_2}{\omega} E_0^2 \hat{e}_z$$

Check

(b) $\underset{\sim}{B} = ?$ : $\quad \nabla \times E + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad \Longrightarrow -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = \nabla \times E$

$$\nabla \times E = \hat{e}_x \left( \underbrace{\frac{\partial E_z}{\partial y}}_{\to 0} - \frac{\partial E_y}{\partial z} \right) + \hat{e}_y \left( \underbrace{\frac{\partial E_x}{\partial z}}_{\to 0} - \underbrace{\frac{\partial E_z}{\partial x}}_{\to 0} \right) + \hat{e}_z \left( \frac{\partial E_y}{\partial x} - \underbrace{\frac{\partial E_x}{\partial y}}_{\to 0} \right)$$

$$\nabla \times E = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = + \hat{e}_x E_0 \cos k_1 x \, k_2 \sin(k_2 z - \omega t) + \hat{e}_z (-) E_0 k_1 \sin k_1 x \cos(k_2 z - \omega t)$$

$$\vec{B} = \int dt \left[ -c E_0 \cos k_1 x \, k_2 \sin(k_2 z - \omega t) \hat{e}_x + c E_0 k_1 \sin k_1 x \cos(k_2 z - \omega t) \hat{e}_z \right]$$

let $x = k_2 z - \omega t$ , $dx = -\omega dt$, , $dt = dx/-\omega$

$$= - \int dx \left[ -\frac{c E_0}{\omega} \cos k_1 x \, k_2 \sin x \, \hat{e}_x + c \frac{E_0}{\omega} k_1 \sin k_1 x \cos x \, \hat{e}_z \right]$$

$$\boxed{\vec{B} = -k_2 \frac{c E_0}{\omega} \cos k_1 x \cos(k_2 z - \omega t) \hat{e}_x - \frac{c E_0 k_1}{\omega} \sin k_1 x \sin(k_2 z - \omega t) \hat{e}_z + \text{Constant}}$$

#5

$bx + c$

(a) solution to unperturbed problem

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2} \qquad n = 1, 2, 3, \ldots$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \qquad n = 2, 4, 6 \ldots$$

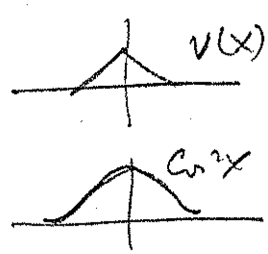
$$= \sqrt{\frac{2}{a}} \cos \frac{n\pi x}{a} \qquad n = 1, 3, 5 \ldots$$

(i) $$\delta E^{(1)} = \int \psi^2 V(x)\, dx$$

$$\propto \int_{-a/2}^{a/2} x \cos^2 x\, dx = 0$$ (integrand is odd)

(ii) $$\delta E \propto \int x \sin^2 x\, dx = 0$$ "

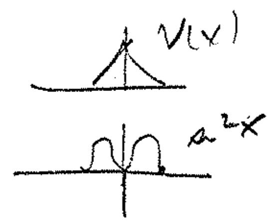
(iii) $$\delta E \propto \int_{-a/2}^{a/2} V(x) \cos^2 x\, dx$$




product is always > 0

$\delta E > 0$

(iv) $$\delta E \propto \int V(x) \sin^2 x\, dx$$




$\delta E > 0$

(v) $$\delta E_m = \sum_{n \neq m} \frac{|\langle m|V|n\rangle|^2}{E_m - E_n} < 0 \quad \text{since } E_n > E_m$$

(b) $\psi = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$ $\quad n = 1, 2, 3, 4, \cdots$

$E = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$

$$\delta E^{(1)} = \int_0^a bx \; \psi^2 \, dx = \frac{2b}{a} \int_0^a \sin^2 \frac{n\pi x}{a} \, x \, dx$$

$$= \frac{2b}{a} \left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} u \sin^2 u \, du$$

$$= \frac{2b}{a} \left(\frac{a}{n\pi}\right)^2 \left[ \frac{u^2}{4} - \frac{u \sin 2u}{4} - \frac{\cos 2u}{8} \right]_0^{n\pi}$$

$$= \frac{2b}{a} \left(\frac{a}{n\pi}\right)^2 \left[ \frac{(n\pi)^2}{4} - \frac{1}{8} + \frac{1}{8} \right]$$

$$= \frac{ab}{2}$$

Ans, #6:

a) Fig. 1 represents radiative decay of the 2p levels;
5 whereas, in Part 2 of the exp't the applied E-field mixes the 2s and 2p levels, causing the exponential decay to have superimposed on it a sinusoidal modulation, at the Lamb shift frequency, neglecting Stark shifts.

b) For Fig 1 $|i\rangle = |2p\rangle e^{-\left(\frac{iE_p}{\hbar} + \gamma/2\right)t}$; and the light intensity is determined by

$|\langle i|z|f\rangle|^2 \propto e^{-\gamma t}$. For Fig. 2

$$|i\rangle = c_p \psi_1 e^{-\left(\frac{iE_p}{\hbar} + \gamma/2\right)t} + c_s \psi_2 e^{-\frac{iE_s t}{\hbar}},$$

~~$|\langle f|z|i\rangle|^2 \propto |$~~ where $\psi_1$ and $\psi_2$ are both mixtures of 2s and 2p eigenfunctions
10 the matrix element $\langle i|z|f\rangle$ will pick out the p-part of $\psi_1$ and $\psi_2$, giving

$\langle i|z|f\rangle \propto c_p' e^{-i\omega_p t - \gamma/2 t} + c_s' e^{-i\omega_s t}$. Assume $c_s'$ and $c_p'$ to be real, and examine

$$|\langle i|z|f\rangle|^2 \propto (c_p')^2 e^{-\gamma t} + (c_s')^2 + (\text{coeff})\, e^{-\gamma/2 t} \cos \Delta\omega t,$$

where $\Delta\omega = \omega_s - \omega_p$. With proper adjustment of coefficients, the data of Fig 2 could be fitted.

c) From Fig. 2 the period of oscillation is ~.9 nsec;
5 $\therefore \Delta\nu_L \approx 1.1\,\text{GHz}$

④ Solutions / E.M.

$$\vec{E} = E_0 \cos k_1 x \cos(k_2 z - \omega t)\, \hat{e}_y$$

ⓐ Phase velocity

constant phase ⟹ $k_2 z - \omega t = \text{Const.}$

$\underline{V = \omega / k_2 \text{, in } + z \text{ direction}}$

ⓑ $\vec{B} = ?$

let $\vec{B} = \text{Re}\left(\vec{B}(r) e^{-i\omega t}\right)$

$\vec{E} = \text{Re}\left(E_0 e^{i(k_2 z - \omega t)} \cos k_1 x\right) \hat{e}_y$ (Done other way after (e) as check)

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

so $\frac{i\omega}{c} B = \nabla \times E$

$$= \hat{e}_x\left(\cancel{\frac{\partial E_z}{\partial y}}^{\,0} - \frac{\partial E_y}{\partial z}\right) + \hat{e}_y\left(\cancel{\frac{\partial E_x}{\partial z}}^{\,0} - \cancel{\frac{\partial E_z}{\partial x}}^{\,0}\right) + \hat{e}_z\left(\frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}}^{\,0}\right)$$

$$\frac{i\omega}{c} B = -\hat{e}_x E_0 \cos k_1 x (i k_2) e^{i k_2 z - \omega t} + \hat{e}_z E_0(-k_1 \sin k_1 x) e^{i k_2 z - \omega t}$$

$$B = -\frac{c E_0}{\omega} k_2 \cos k_1 x\, e^{i k_2 z - \omega t} \hat{e}_x + \frac{i c E_0}{\omega} k_1 \sin k_1 x\, e^{i k_2 z - \omega t} \hat{e}_z$$

take real part

$$\vec{B} = -\frac{c E_0 k_2}{\omega} \cos k_1 x \cos(k_2 z - \omega t)\hat{e}_x + \frac{c E_0 k_1}{\omega} \sin k_1 x \sin(k_2 z - \omega t)\hat{e}_z$$

ⓒ $\nabla \times B = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \hat{e}_y\left[\frac{\omega E_0}{c} \cos k_1 x \sin(k_2 z - \omega t)\right]$$

$$\nabla \times B\Big|_y = \hat{e}_y\left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right]$$

$$= \hat{e}_y\left[+\frac{c E_0 k_2^2}{\omega} \cos k_1 x \sin(k_2 z - \omega t) + \frac{c E_0 k_1^2}{\omega} \cos k_1 x \sin(\;)\right]$$

compare these two : $c\frac{k_2^2}{\omega} + \frac{c k_1^2}{\omega} = \frac{\omega}{c}$

$$\underline{k_2^2 = \frac{\omega^2}{c^2} - k_1^2}$$

Soltns

(d) Wherever $\underset{\sim}{E}$ has a node for all times, i.e. $\left\{ \begin{array}{l} k_1 x = n \cdot \frac{\pi}{2} \\ x = \frac{\pi}{2} \frac{n}{k_1} \end{array} \right.$

with n odd integer ; yet also need

$$k_2^2 = \frac{\omega^2}{c^2} - k_1^2$$

$$\text{with } k_2^2 > 0$$

$$k_1 < \frac{\omega}{c}$$

(e) Flux

at $x = y = z = t = 0$

$$\vec{E} = E_0 \hat{e}_y$$

$$\vec{B} = + \frac{k_2 E_0 c}{\omega} \hat{e}_x$$

$$\underset{\sim}{S} = \text{Power/area} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$= + \frac{c^2}{4\pi} \frac{k_2}{\omega} E_0^2 \hat{e}_z$$

Check

(b) $\underset{\sim}{B} = ?$ : $\underset{\sim}{\nabla} \times \underset{\sim}{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \implies -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \underset{\sim}{\nabla} \times \underset{\sim}{E}$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = \hat{e}_x \left( \underbrace{\cancel{\frac{\partial E_z}{\partial y}}}_{\to 0} - \frac{\partial E_y}{\partial z} \right) + \hat{e}_y \left( \underbrace{\cancel{\frac{\partial E_x}{\partial z}}}_{\to 0} - \underbrace{\cancel{\frac{\partial E_z}{\partial x}}}_{\to 0} \right) + \hat{e}_z \left( \frac{\partial E_y}{\partial x} - \underbrace{\cancel{\frac{\partial E_x}{\partial y}}}_{\to 0} \right)$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = + \hat{e}_x E_0 \cos k_1 x \, k_2 \sin(k_2 z - \omega t) + \hat{e}_z (-) E_0 k_1 \sin k_1 x \cos(k_2 z - \omega t)$$

$$\vec{B} = \int dt \left[ -c E_0 \cos k_1 x \, k_2 \sin(k_2 z - \omega t) \hat{e}_x + c E_0 k_1 \sin k_1 x \cos(k_2 z - \omega t) \hat{e}_z \right]$$

let $x = k_2 z - \omega t$ , $dx = -\omega dt$ , $dt = dx/-\omega$

$$= \int \frac{dx}{\omega} \left[ -\frac{c E_0}{\omega} \cos k_1 x \, k_2 \sin x \, \hat{e}_x + c \frac{E_0}{\omega} k_1 \sin k_1 x \cos x \, \hat{e}_z \right]$$

$$\boxed{\vec{B} = -k_2 \frac{c E_0}{\omega} \cos k_1 x \cos(k_2 z - \omega t) \hat{e}_x + \frac{c E_0 k_1}{\omega} \sin k_1 x \sin(k_2 z - \omega t) \hat{e}_z + \text{Constant}}$$

Solution ③ (Graduate Stat. Mech).

(a) $\mathcal{H} = p_1^2/2m_1 + p_2^2/2m_2 + \frac{s}{2}(\vec{r}_1 - \vec{r}_2)^2$ (For 1 pair)

(10 pts) $Z = \int e^{-\beta \mathcal{H}} d^3r_1 d^3r_2 d^3p_1 d^3p_2$ $\quad (\beta = 1/kT)$

$$= \int_0^\infty 4\pi p_1^2 dp_1 e^{-\beta p_1^2/2m_1} \int_0^\infty 4\pi p_2^2 e^{-\beta p_2^2/2m_2} \int d^3r_1 d^3r_2 e^{-\beta \frac{s}{2}(r_1 - r_2)^2}$$

let $r = r_1 - r_2$, $x = \frac{r_1 + r_2}{2}$

$$\left(\frac{4\pi\sqrt{\pi}(2m_1)^{3/2}}{4\beta^{3/2}}\right) \left(\frac{4\pi\sqrt{\pi}(2m_1)^{3/2}}{4\beta^{3/2}}\right) \underbrace{\int d^3x}_{V} \int d^3r\, e^{-\frac{\beta s}{2}r^2}$$

$$\left(\frac{4\pi\sqrt{\pi}\, 2^{3/2}}{4\beta^{3/2}s^{3/2}}\right)$$

give

$$\boxed{\int_0^\infty x^{2a} e^{-r^2x^2} dx = \frac{1\cdot 3\cdot 5 \cdots (2a-1)}{2^{a+1}\, r^{2a+1}}}$$

$$\underset{\text{(one pair)}}{Z_1} = \left(\frac{2\pi m_1}{\beta}\right)^{3/2} \left(\frac{2\pi m_2}{\beta}\right)^{3/2} V \left(\frac{2\pi}{\beta s}\right)^{3/2}, \quad Z_{all} \propto (Z_1)^N$$

$$\left(\ln Z_1 = \frac{3}{2}\ln\frac{m_1}{\beta} + \frac{3}{2}\ln\frac{m_2}{\beta} + \frac{3}{2}\ln\frac{1}{\beta s} + \ln V\right)$$

(b) (6 pts) $\langle \mathcal{H} \rangle = -\frac{\partial}{\partial\beta} \ln Z \Big|_{N,V}$

$\ln Z = -\frac{9}{2}\ln\beta\, N + \text{const}$

(3 pts) $\langle \mathcal{H} \rangle = \frac{9}{2}\, 1/\beta\, N = \underline{\underline{\frac{9}{2} kT N}}$

---

(2 pts) $C_V = \frac{\partial \langle \mathcal{H} \rangle}{\partial T}\Big|_{N,V} = \frac{\partial E}{\partial T}\Big|_{N,V} = \frac{9}{2} k N$

(2 pts) At first this $\mathcal{H}$ looks like the one for a diatomic molecule, yet there we expect $C_V = 7/2\, k$. The difference is due to the molecule having a "spring" with nonzero unstretched length and ∴ only 1 degree of vibrational freedom. For a zero length spring, there are 3 degrees of freedom ∴ 2 more $\frac{1}{2}k$'s.

(c) (2 pts) $\frac{\partial Z_1}{\partial s} = \int d^3 r_1 r_2 p_1 p_2 \left[ -\frac{\beta}{2} (\underset{\sim}{r_1} - \underset{\sim}{r_2})^2 \right] e^{-\beta \mathcal{H}} = -\frac{\beta}{2} \langle (r_1 - r_2)^2 \rangle Z$

so $$-\frac{\beta}{2} \langle (r_1 - r_2)^2 \rangle = \frac{1}{Z} \frac{\partial Z}{\partial s} = \frac{\partial \ln Z}{\partial s} = -\frac{3}{2s} N$$

$$\underline{\frac{1}{2} s \langle (\vec{r}_1 - \vec{r}_2)^2 \rangle = \frac{3}{2} kT N}$$

(d) (2 pts). Since the momentum and space integrals seperate, the two overages ae statistically independent, thus

$$\langle (\vec{p}_1)^2 (x_1 - x_2)^{24} \rangle = \langle p_1^2 \rangle \langle (x_1 - x_2)^{24} \rangle \qquad \underline{So}$$

$\underline{\text{answer} = 0}$