#24

PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #24

April 3, 1976

General Instructions

This Comprehensive Examination for Spring 1976 (#24) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers, except pencil and bluebook, on the floor.

Answer all questions. Each question is worth 20 points. Show all work so that part credit may be given where justified.

1. (A) Describe the essential features of one experiment which gives direct evidence for quantization in each of the following cases:

   (a) The energy of monochromatic radiation.

   (b) The component of angular momentum in a fixed direction.

   (c) The excitation energy of an atom.

   (d) The number of vortex lines in a superfluid.

   In each of these, contrast the actual observation with predictions of classical physics.

   (B) Describe the essential features of an experiment which establishes the wave nature of electrons, and one which establishes the particle nature of light. How are each of these reconciled with their respective classical descriptions?

2. A π^0 meson with kinetic energy in the laboratory frame that is equal to its rest energy is moving in the x-direction. It decays into two photons which are emitted perpendicular to the direction of motion in the rest frame of the π^0. Find the energies and directions of motion of the two photons in the laboratory frame.

3. The polarization state of plane polarized light (propagating, say, in the z-direction) can be represented by a normalized two-component vector

$$|\psi\rangle \qquad \begin{pmatrix} E_x \\ E_y \end{pmatrix} \left(E_x{}^2 + E_y{}^2\right)^{-\frac{1}{2}}$$

If a photon from an incident beam passes through an x-oriented polarizer it emerges with x-polarization. If it passes a y-oriented polarizer the photon emerges with y-polarization. With such polarizers the degree of x and y polarization of the beam can be determined.

Any such dynamical observable (e.g. the transparency or opacity of the polarizer to a photon) is represented in quantum mechanics by an Hermitian operator whose eigenvalues are the measurable but unpredictable results of any single measurement.

Consider the two matrix operators

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) Find the eigenvalues and normalized eigenfunctions of these operators.

(b) Discuss the interpretation of the eigenvalues and eigenvectors in terms of the measurement made by a polarizer placed in the path of a plane polarized light beam.

(c) Show that for an incident light beam we may write

$$\langle P_x\rangle = \cos^2\phi \qquad\qquad \langle P_y\rangle = \sin^2\phi$$

(d) Show that for an incident light beam the uncertainty in the degree of x-polarization is

$$\langle(\Delta P_x)^2\rangle = \cos^2\phi \, \sin^2\phi$$

where ϕ is some suitably defined angle.

4. Three electrons are confined to the vertices of an equilateral triangle and they interact with each other via the magnetic dipole-dipole interaction

$$H_{int} = \frac{\lambda}{2} \sum_{i,j} \vec{\mu}_i \vec{\mu}_j \quad (i \neq j)$$

(a) Give the possible values of total spin of the system and the degeneracy of each total spin state.

(b) Find the energy of each of these spin states.

(c) Deduce the partition function of a collection of N such systems. (Assume that they do not interact with each other.)

5. The ground state of the field free (magnetic field free and radiation field free) hydrogen atom is split into two levels by the hyperfine interaction

$$H_{HyP} = a\, \vec{I}\cdot\vec{J}$$

with an energy separation $\Delta E \cong 1420$ MHZ. Here $\vec{I}$ is the proton spin angular momentum operator, $\vec{J}$ is the total electron angular momentum operator with $\vec{J} = \vec{L} + \vec{S}$, $\vec{S}$ being the electron spin operator and $\vec{L}$ the electron orbital angular momentum operator. The proton spin angular momentum quantum number is $I = 1/2$, and for the hydrogen atom ground state $L = 0$ so that the quantum number J is also 1/2.

(a) Find the wavefunction, in terms of the electron and proton spin wavefunctions for each of the states making up the hydrogen ground state. For each state list the good quantum numbers.

(b) For each state find the energy in terms of E_0, the energy of the atom in the limit $a \to 0$.

6. A mass m is attached to the rim of a wheel with a massless, inextensible rod. The rod has length L and the wheel has radius R. The wheel is rotated in the horizontal plane with constant angular velocity ω.

(a) Write the Lagrangian for the rotating mass.

(b) If the equilibrium angle the rod makes with the horizontal is 45°, find the frequency of small oscillations about that position.

(1)

Possibilities

A - 1 - photo electric effect
2 - Stern - Gerlach exp.
3 - Franck - Hertz exp.

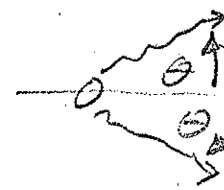
B 1 Diffraction of electron - Davisson - Germer, Thom
2 Compton - scattering

(2)

Solution

(a) In meson rest frame

$$E_{photon} = \frac{m_{\pi^0}}{2}$$



(conservation of momentum $\Rightarrow E_{\gamma_1} = E_{\gamma_2}$)

velocity of meson rest frame relative to laboratory:

$$K = m \qquad \therefore E = 2m = \frac{m}{\sqrt{1-\beta^2}}$$

$$\therefore \beta^2 = 3/4 , \quad \beta = \tfrac{1}{2}\sqrt{3}$$

$$P_\pi = \sqrt{E^2 - m^2} = \sqrt{4m^2 - m^2} = \sqrt{3}\, m$$

by symmetry each photon has $P_x = \frac{\sqrt{3}}{2} m$

$$P_y = \tfrac{1}{2} m$$

$$\therefore p = p_x^2 + p_y^2 = \left(\tfrac{3}{4} + \tfrac{1}{4}\right) m = m$$

$$= \boxed{E = m}$$

$$\tan\theta = \frac{P_x}{P_y} = \sqrt{3}$$

$$\boxed{\theta = 30^\circ}$$

(3)

(a) $P_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix} = p_x \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix}$$

$$\det \begin{pmatrix} 1-p_x & 0 \\ 0 & -p_x \end{pmatrix} = 0 \qquad \text{det. eigenvalues}$$

$$(1-p_x)(p_x) = 0$$

$$p_x = 1 \quad p_x = 0$$

For eigenvectors

$$p_x = 1 \quad \pi_1^x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad p_x = 0 \quad \pi_0^x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

eigenvalues $= 0, 1$

For eigenvectors

$$p_y = 1 \quad \pi_1^y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad\qquad p_y = 0 \quad \pi_0^y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b) For any photon, the ~~polarizer~~ photon will ~~either~~ either pass the x-polarizer ($p_x = 1$) or be rejected by the x polarizer ($p_x = 0$).
A passed photon has the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ projected out by the measurement.

3

(c) $\langle P_x \rangle = \langle \psi | P_x | \psi \rangle$

Expand

$$|\psi\rangle = p_1^x \langle \Pi_1 | \psi \rangle + p_0^x \langle \Pi_0 | \psi \rangle$$

$$\langle P_y \rangle = \langle \psi | P_y | \psi \rangle$$

$$|\psi\rangle = p_1^y \langle \Pi_1^y | \psi \rangle + p_0^y \langle \Pi_0^y | \psi \rangle$$

But $P_x + P_y = I$

So $\langle \psi | P_x + P_y | \psi \rangle = 1$

and

$$|\langle \Pi_1^x | \psi \rangle|^2 + |\langle \Pi_1^y | \psi \rangle|^2 = 1$$

choose $|\langle \Pi_1^x | \psi \rangle|^2 = \cos^2 \varphi$

etc.

(d)

$$\langle \psi | (P_x - \langle P_x \rangle)^2 | \psi \rangle$$

$$= \langle \psi | P_x^2 - 2 P_x \langle P_x \rangle + \langle P_x \rangle^2 | \psi \rangle$$

$$= \langle P_x \rangle - 2 \langle P_x \rangle^2 + \langle P_x \rangle^2$$

$$= \cos^2 \phi \, (1 - \cos^2 \phi)$$

$$= \cos^2 \varphi \, \sin^2 \varphi$$

4) Statistical Physics – Solution

(a)

$$H = \lambda'(\vec{\mu}_1\cdot\vec{\mu}_2 + \vec{\mu}_1\cdot\vec{\mu}_3 + \vec{\mu}_2\cdot\vec{\mu}_3)$$

$$= \lambda(\vec{\sigma}_1\cdot\vec{\sigma}_2 + \vec{\sigma}_1\cdot\vec{\sigma}_3 + \vec{\sigma}_2\cdot\vec{\sigma}_3)$$

(b)

$$\vec{S} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3)$$

$$\vec{S}^2 = \frac{1}{4}\big(\vec{\sigma}_1\cdot\vec{\sigma}_2 + \vec{\sigma}_2\cdot\vec{\sigma}_1 + \vec{\sigma}_1\cdot\vec{\sigma}_3 + \vec{\sigma}_3\cdot\vec{\sigma}_1 + \vec{\sigma}_2\cdot\vec{\sigma}_3 + \vec{\sigma}_3\cdot\vec{\sigma}_2 + \vec{\sigma}_1^{\,2} + \vec{\sigma}_2^{\,2} + \vec{\sigma}_3^{\,2}\big)$$

$$= \frac{1}{4}\left[2\frac{H}{\lambda} + 9\right]$$

$$= \frac{H}{2\lambda} + \frac{9}{4}$$

$$\boxed{H = 2\lambda\left(\vec{S}^2 - \frac{9}{4}\right)}$$

$S = 3/2$

$$E = 2\lambda\left[\frac{3}{2}\cdot\frac{5}{2} - \frac{9}{4}\right] = 3\lambda$$

$S = \frac{1}{2}$ $\quad E = 2\lambda\left[\frac{1}{2}\cdot\frac{3}{2} - \frac{9}{4}\right] = -3\lambda$

$S = \frac{3}{2}$ degeneracy $= 4$ $\quad (2S+1)$

$S = \frac{1}{2}$ degeneracy $= 4$ $\quad (2S+1)\times 2$

$$Z = \sum e^{-E_n/kT} = 4e^{-3\lambda/kT} + 4e^{3\lambda/kT}$$

$$= 8\cosh\frac{3\lambda}{kT}$$

(4)

$$Z = 8^N \cosh^N \frac{3\lambda}{kT}$$

$$F = -kT \ln Z = -NkT \ln\left(8\cosh\frac{3\lambda}{kT}\right)$$

$$S = -\frac{\partial F}{\partial T} = +Nk \ln\left(8\cosh\frac{3\lambda}{kT}\right) + NkT \frac{1}{8\cosh\frac{3\lambda}{kT}} \frac{d}{dT}\left(8\cosh\frac{3\lambda}{kT}\right)$$

$$= Nk(\ln 8) + Nk \ln\left(\frac{e^{\frac{3\lambda}{kT}} + e^{-\frac{3\lambda}{kT}}}{2}\right) + \frac{NkT}{\cosh\frac{3\lambda}{kT}}\left(\sinh\frac{3\lambda}{kT}\right)\left(-\frac{3\lambda}{kT^2}\right)$$

$$= Nk \ln 8 - Nk \ln 2 + Nk \ln\left[e^{\frac{3\lambda}{kT}}\left(1+e^{-\frac{6\lambda}{kT}}\right)\right] - \frac{3N\lambda}{T}\tanh\frac{3\lambda}{kT}$$

$$= Nk \ln 4 + Nk\left(\frac{3\lambda}{kT}\right) + Nk \ln\left(1+e^{-\frac{6\lambda}{kT}}\right) - \frac{3N\lambda}{T}\frac{3\lambda}{kT}$$

$$= Nk \ln 4 + \frac{3N\lambda}{T} - \frac{9N\lambda^2}{kT^2} + Nk\left(1 - \frac{6\lambda}{kT}\right)$$

$$\boxed{S = Nk\left(1 + \ln 4 - \frac{3\lambda}{kT} - \frac{9\lambda^2}{(kT)^2}\right)}$$

1

5

$H = H^0 + a(\vec{I}\cdot\vec{J})$ where $H^0\psi = E_0\psi$

Now: $\vec{I}\cdot\vec{J} = \vec{I}\cdot\vec{S}$ as $\vec{L}=0$

Defn $\vec{I} = \hbar/2\,\vec{\sigma}^p \quad \vec{S} = \hbar/2\,\vec{\sigma}^e$

$\vec{I}\cdot\vec{S} = \frac{\hbar^2}{4}\left(\sigma_x^p\sigma_x^e + \sigma_y^p\sigma_y^e + \sigma_z^p\sigma_z^e\right)$

and the $\alpha^p\beta^p$ & $\alpha^e\beta^e$ are spin functions

We can try spin states (1) $\alpha^p\alpha^e$, (2) $\alpha^p\beta^e$, (3) $\beta^p\alpha^e$, (4) $\beta^p\beta^e$

Matrix elements

	(1)	(2)	(3)	(4)
(1)	1	0	0	0
(2)	0	-1	2	0
(3)	0	2	-1	0
(4)		0	0	1

since $\vec{I}\cdot\vec{S}\,|\alpha^p\alpha^e\rangle = \frac{\hbar^2}{4}\alpha^p\alpha^e$

$\vec{I}\cdot\vec{S}\,|\alpha^p\beta^e\rangle = \frac{\hbar^2}{4}\left(2\beta^p\alpha^e - \alpha^p\beta^e\right)$

$\vec{I}\cdot\vec{S}\,|\beta^p\alpha^e\rangle = \frac{\hbar^2}{4}\left(2\alpha^p\beta^e - \beta^p\alpha^e\right)$

$\vec{I}\cdot\vec{S}\,|\beta^p\beta^e\rangle = \frac{\hbar^2}{4}\left(\beta^p\beta^e\right)$

[illegible]

[illegible]

Not really necessary

5

When the (2)-(3) sub matrix is diagonalized the diagonal elements are 1, -3 and it can be easily seen the eigenfunctions but have the relation for 1 that $a^{(1)} = b^{(1)}$ and for -3 that $a^{(-3)} = -a^{(-3)}$ where $|a^{(1)}|^2 + |b^{(1)}|^2 = 1 = |a^{-3}|^2 + |b^{-3}|^2$

$$\psi_1 = \alpha^p \alpha^e \qquad E = E_0 + 1/4\,\Delta E$$
$$\psi_2 = \frac{1}{\sqrt{2}}(\alpha^p\beta^e + \beta^e\alpha^-) \qquad E = E_0 + 1/4\,\Delta E$$
$$\psi_3 = \beta^p\beta^e \qquad E = E_0 + 1/4\,\Delta E$$
Triplet

$$\psi_4 = \frac{1}{\sqrt{2}}(\alpha^p\beta^e - \beta^p\alpha^e) \qquad E = E_0 - 3/4\,\Delta E$$ Singlet

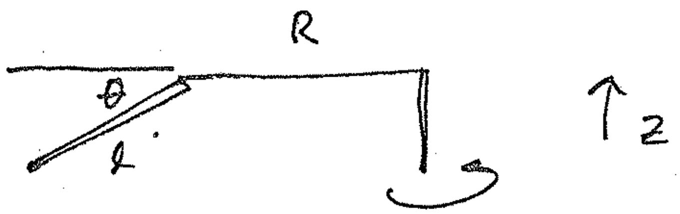
The quantum no. & not the

ψ	m_s	m_I	m_F	E
1	½	½	1	
2	not good	Q. no.	0	
3	-½	-½	-1	
4	not good	Q no.	0	

Seems tough way to solve

Since $\vec{I}\cdot\vec{J} = \frac{F^2 - I^2 - J^2}{2}$ | if and (3) F, m_F

(6)



$$x = (R + l\cos\theta)\cos\omega t$$
$$y = (R + l\cos\theta)\sin\omega t$$
$$z = -l\sin\theta$$

$$\dot{x} = -\omega(R + l\cos\theta)\sin\omega t \quad - \dot{\theta} l \sin\theta \cos\omega t$$
$$\dot{y} = \omega(R + l\cos\theta)\cos\omega t \quad - \dot{\theta} l \sin\theta \sin\omega t$$
$$\dot{z} = -\dot{\theta} l \cos\theta$$

$$(\dot{x})^2 = \omega^2 (R + l\cos\theta)^2 \sin^2\omega t \quad + \dot{\theta}^2 l^2 \sin^2\theta \cos^2\omega t$$
$$+ 2\omega(R + l\cos\theta)\dot{\theta} l \sin\theta \sin\omega t \cos\omega t$$

$$(\dot{y})^2 = \omega^2 (R + l\cos\theta)^2 \cos^2\omega t \quad + \dot{\theta}^2 l^2 \sin^2\theta \sin^2\omega t$$
$$- 2\omega(R + l\cos\theta)\dot{\theta} l \sin\theta \cos\omega t \sin\omega t$$

$$\dot{z}^2 = \dot{\theta}^2 l^2 \cos^2\theta$$

$$T = \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] = \frac{1}{2} m \left[\omega^2 (R + l\cos\theta)^2 + \dot{\theta}^2 l^2\right]$$

$$V = mgl(1 - \sin\theta)$$

$$\mathcal{L} = \frac{1}{2} m \left[\omega^2 (R + l\cos\theta)^2 + \dot{\theta}^2 l^2\right] - mgl(1 - \sin\theta)$$

(6)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = m\ell^2\ddot\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m\omega^2\ell(R+\ell\cos\theta)\sin\theta + mg\ell\cos\theta$$

EQ. of Motion

$$m\ell^2\ddot\theta + m\omega^2\ell(R+\ell\cos\theta)\sin\theta - mg\ell\cos\theta = 0$$

Position of Equilibrium

$$\omega^2(R+\ell\cos\theta)\sin\theta = g\cos\theta$$

Expand about $\theta = \theta_0 = \frac{\pi}{4}$

$$m\ell^2\ddot\theta + \left\{m\omega^2\ell\left[(R+\ell\cos\theta_0)\cos\theta_0 - \ell\sin^2\theta_0\right] + mg\ell\sin\theta_0\right\}(\theta-\theta_0) = 0$$

$\theta_0 = \frac{\pi}{2}$

$$m\ell^2\ddot\theta + \left\{m\omega^2\ell\left[\left(R+\frac{\ell\sqrt2}{2}\right)\frac{\sqrt2}{2} - \frac{\ell}{2}\right] + mg\frac{\ell\sqrt2}{2}\right\}\Delta\theta = 0$$

$$m\ell^2\ddot\theta + \left(m\omega^2\ell R\frac{\sqrt2}{2} + mg\ell\frac{\sqrt2}{2}\right)\Delta\theta = 0$$

$$m\ell^2\ddot\theta + \frac{m\ell\sqrt2}{2}(R\omega^2+g)\Delta\theta$$

But for 45° equil.

$$\omega^2\left(R+\frac{\ell\sqrt2}{2}\right) = g$$