PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #23

January 17, 1976

General Instructions

This Comprehensive Examination for Winter 1976 (#23) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers, except pencil and bluebook, on the floor.

Some information you may find useful:

$$\int \sin^2 x\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$$

$$\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$$

$$\text{for } m^2 \neq n^2$$

$$\int_0^\infty dx\,\frac{\sqrt{x}}{e^x - 1} = 1.3\sqrt{\pi}$$

1. A particle of mass m moves in a one dimensional box of width L and potential V such that $V=0$ for $0 \leq x \leq L$ and $V=\infty$ otherwise. At time $t=0$ the unnormalized wave function of the particle is

$$\psi_{(x,0)} = \sin \frac{x\pi}{L} + 1/2 \sin \frac{3x\pi}{L}$$

(5) (a) Find the normalized wave function at all times $t>0$.

(5) (b) What are the possible outcomes of a measurement of the energy of the particle? What are the probabilities of observing each value? Calculate the average value of the energy of the particle, $\langle E \rangle$.

(10) (c) What is the minimum time that must elapse before the probability of finding the particle in the interval x , $x+dx$ returns to that of the initial state?

2. A gas of molecules having a number density n and temperature T is enclosed in a container whose dimensions are large compared to the mean free path of the molecules. A small hole of area A is punched in a side of the container and the gas is allowed to escape slowly into an evacuated space.

The unnormalized speed distribution of the gas inside the container is

$$N(v)dv = e - \frac{mv^2}{2K_BT} v^2 dv$$

where K_B is Boltzmann's constant and m is the molecular mass.

(4) (a) What is the total number of molecules per second escaping the container?

(4) (b) What is the speed distribution, $N_e(v)dv$, of the escaping gas?

(4) (c) Find the mean speed, RMS speed, and the most probable speed of the gas molecules in the container.

(4) (d) Find the mean speed, RMS speed, and the most probable speed of the gas molecules that escape the container.

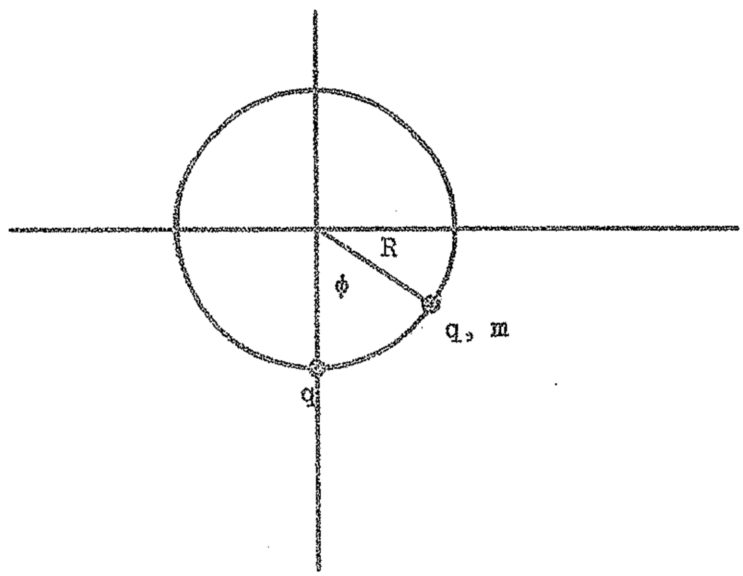
(4) (e) Describe briefly, but critically, an experiment to determine the distribution of speeds for the molecules of a gas at a fixed temperature T .

3. A small bead of charge q is fixed at the bottom of a ring which lies in a vertical plane. The radius of the ring is R. A second bead of like charge q and mass m slides without friction along the ring.

(5) (a) Find the net torque, taken about the center of the ring, that acts on the second bead. Express the result as a function of the angle ϕ.

(5) (b) Find the equilibrium position ϕ_0, of the second bead.

(10) (c) Find the frequency of small amplitude oscillations made by the second bead about the equilibrium angle, ϕ_0. (Assume $q^2 \ll mgR^2$ so that the second bead lies in the quadrant shown. g is the acceleration due to gravity.)



4. An ideal transmission line consists of two (or more) perfect conductors of arbitrary uniform cross-section that extend, say, in the z-direction.

(3) (a) Write an expression that describes a monochromatic TEM wave that propagates along the transmission line in the positive z-direction.

(7) (b) Starting from Maxwell's Equations show that the electric and magnetic fields of this TEM wave satisfy the same equations as do <u>electrostatic</u> and <u>magnetostatic</u> fields.

(7) (c) If the electric and magnetic fields were independent of time then their values would be independent of each other. Show that the electric and magnetic fields are dependent for the monochromatic TEM wave that propagates in the positive z-direction.

(3) (d) In the practical operation of a transmission line the ratio of the electric to magnetic field (and accordingly voltage to current) can take on any value. Why is this so?

5. Many models of liquid He^4 treat the He^4 atoms as an ideal spin zero Bose gas. This ideal Bose gas undergoes a phase transition called the Bose-Einstein condensation at a temperature T_c which is characterized by a macroscopic number of particles just beginning to appear in the ground state of the system. Above T_c the particles all occupy excited states of the system. At $T=0$ all the particles are finally in the ground state.

(5) (a) Show that for the ideal Bose system the chemical potential, μ, satisfies the inequality $\mu \leq 0$.
(Assume the ground state energy of the particles is zero.)

(5) (b) Show that in the limit $T\to 0$ the chemical potential for a macroscopic Bose gas satisfies the equality

$$e^{\frac{\mu}{K_B T}} \cong 1 - \frac{1}{N}$$

where N is the number of gas particles and K_B is Boltzmann's constant.

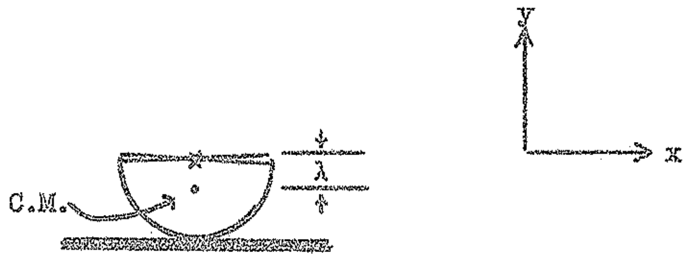
(3) (c) Photons, phonons, magnons, plasmons, are also Bosons but they do not undergo a Bose-Einstein condensation. Why not?

(7) (d) Show that the Bose-Einstein condensation temperature for the ideal Bose gas is given by

$$T_c = \frac{2\pi\hbar^2}{M} \left(\frac{n}{2.6}\right)^{2/3}$$

where M is the mass of the particles and n is the number density of the particles.

6. Half a right circular cylinder (cut on the cylindrical axis) is allowed to rock on a horizontal surface.



The radius of the half cylinder is R, its length is L, its mass is M and its moment of inertia about an axis parallel to the cylindrical axis and passing through the center of mass is I. λ is the distance from the flat surface to the C.M.

Find the Lagrangian for the rocking cylinder in the case

(5) (a) It rocks on an essentially frictionless horizontal surface.

(5) (b) It rocks without slipping on the horizontal surface.

(5) (c) Find for case (a) above, the frequency of small amplitude rocking oscillations.

(5) (d) Find for case (b) above, the frequency of small amplitude rocking oscillations.

#1-1 Q.M. (Undergraduate) <u>Solution</u>

$$\psi(x,0) = \sin\frac{\pi x}{L} + \frac{1}{2}\sin\frac{3\pi x}{L}$$

Normalize

$$\psi(x,0) = C\left(\sin\frac{\pi x}{L} + \frac{1}{2}\sin\frac{3\pi x}{L}\right)$$

$$\int \psi^*\psi = \int_0^L C^2\left(\sin^2\frac{\pi x}{L} + \sin\frac{\pi x}{L}\sin\frac{3\pi x}{L} + \frac{1}{4}\sin^2\frac{3\pi x}{L}\right)dx$$

Given: $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$

$$\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \qquad m^2 \neq n^2$$

$u = \frac{\pi x}{L} \quad du = \frac{\pi}{L}dx$

$$\frac{1}{C^2} = \frac{L}{\pi}\int_0^{\pi}\sin^2 u\,du + \left[\frac{\sin\frac{-2\pi x}{L}}{2\left(-\frac{2\pi}{L}\right)} - \frac{\sin\frac{4\pi x}{L}}{2\left(\frac{4\pi}{L}\right)}\right]_0^L + \frac{1}{4}\frac{L}{3\pi}\int_0^{3\pi}\sin^2 u\,du$$

$$= \frac{L}{\cancel{\pi}}\frac{\cancel{\pi}}{2} + \frac{1}{4}\frac{L}{\cancel{3\pi}}\frac{\cancel{3\pi}}{2} = \frac{5L}{8}$$

Energy eigenvalues $\quad \sin k_1 x = \sin\frac{\pi x}{L}$

$$k_1 = \frac{\pi}{L}$$

free particle $\quad E_1 = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}$

$\# 1-2$

expand on complete eigenstates.

$\sin k_3 x = \frac{3\pi x}{L}$

$k_3 = 3\pi/L$

$$E_3 = \frac{\hbar^2 k_3^2}{2m} = \frac{9\hbar^2\pi^2}{2mL^2} = 9\hbar\omega$$

$$\therefore \boxed{\psi(x,t) = \sqrt{\frac{8}{5L}}\left[\sin\frac{\pi x}{L}\, e^{-i\omega t} + \frac{1}{2}\sin\frac{3\pi x}{L}\, e^{-9i\omega t}\right]}$$

(c) $E_1 = \frac{\hbar^2\pi^2}{2mL^2}$ $\quad$ Prob $= (1)^2$ (unnormalized)

E_2 : Prob $= 0$

$E_3 = 9\frac{\hbar^2\pi^2}{2mL^2}$ $\quad$ Prob $= \left(\frac{1}{2}\right)^2$ $\quad$ "

$E_4, E_5 \ldots\ldots$ $\quad$ Prob $= 0$

$\therefore$ prob of $E_1 = \frac{1}{1.25} = 80\%$

prob. of E_3 — 20%

$$\langle E\rangle = .8E_1 + .2E_3 = \hbar\omega(.8 + (9)(.2))$$
$$= \boxed{2.6\,\hbar\omega}$$

$\hbar\omega = \frac{\hbar^2\pi^2}{2mL^2}$

$\frac{13}{\ldots}\frac{\hbar^2\pi^2}{\ldots}$

#1-3

(c)

$$\psi^*(x,0)\,\psi(x,0) = P(x,0) = \frac{8}{5L}\left[\sin^2\frac{\pi x}{L} + \sin\frac{\pi x}{L}\sin\frac{3\pi x}{L} + \frac{1}{4}\sin^2\frac{3\pi x}{L}\right]$$

$$\psi^*(x,t)\,\psi(x,t) = P(x,t) = \frac{8}{5L}\left[\sin\frac{\pi x}{L}e^{i\omega t} + \frac{1}{2}\sin\frac{3\pi x}{L}e^{9i\omega t}\right]$$

$$\times\left[\sin\frac{\pi x}{L}e^{-i\omega t} + \frac{1}{2}\sin\frac{3\pi x}{L}e^{-9i\omega t}\right]$$

$$P(x,t) = \frac{8}{5L}\left[\sin^2\frac{\pi x}{L} + \frac{1}{4}\sin^2\frac{3\pi x}{L} + \frac{1}{2}\sin\frac{\pi x}{L}\sin\frac{3\pi x}{L}\left(e^{-8i\omega t} + e^{+8i\omega t}\right)\right]$$

$$\left[\quad " \quad " \quad " \quad (2\cos 8\omega t)\quad\right]$$

∴ returns to original probability density when

$$\cos 8\omega t = 1$$

$$8\omega t = 2\pi$$

$$\boxed{t = \frac{\pi}{4\omega}}$$

$$\omega = \frac{\hbar\pi^2}{2mL^2}$$

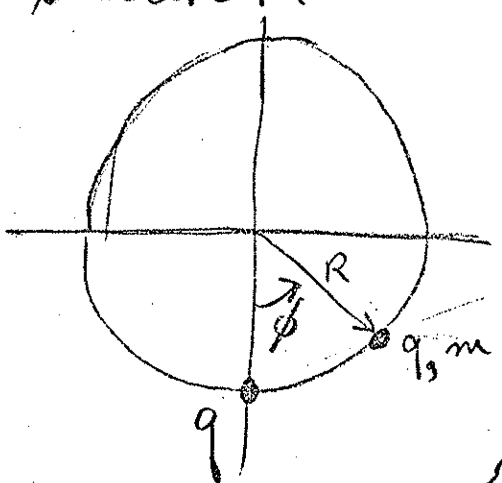
$$E_3 - E_1 = 9\hbar\omega - \hbar\omega = 8\hbar\omega$$

$$\frac{2\pi\hbar}{\ldots\hbar\omega} =$$

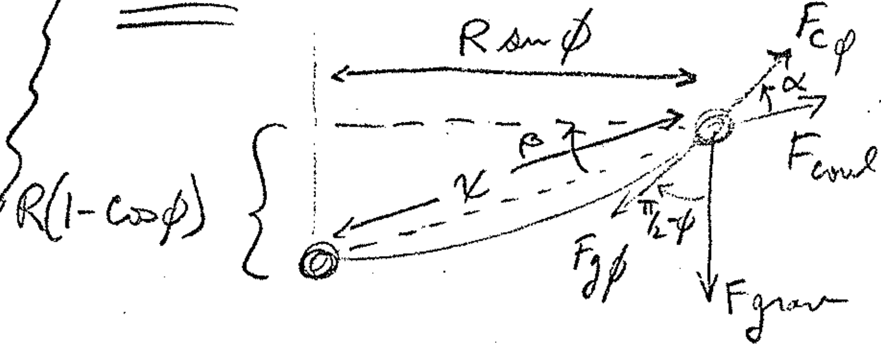
#2-1

Graduate Dynamics

A small bead of charge q is fixed at the bottom of a ring of radius R, which lies in the vertical plane. A second bead of charge q and mass m slides without friction along the ring. ~~which is fixed in a vertical plane~~ (a) Find the equilibrium position, ϕ_0, of the second bead and the frequency for small oscillations about the equilibrium position. (Assume the equilibrium position to be in the quadrant shown.)



Solution:



Since there is no friction, equilibrium occurs when the ϕ-components of F_{grav} + F_{coul} balance

$$x = \sqrt{(R\sin\phi)^2 + R^2(1-\cos\phi)^2} = R\sqrt{2-2\cos\phi}$$

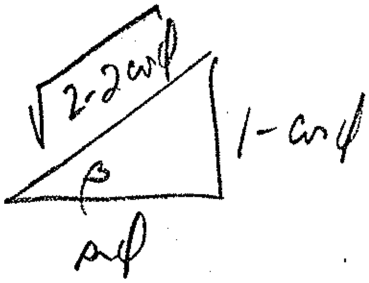
$$F_{g\phi} = mg\sin\phi$$

$$\alpha+\beta+\pi/2-\phi = \pi/2$$

$$F_{c\phi} = \frac{q^2}{x^2}\cos\alpha$$

$$\therefore \alpha = \phi-\beta$$

#2-2



$$\cos\beta = \frac{\sin\phi}{\sqrt{2-2\cos\phi}} = \frac{\sin\phi}{2\sin\phi/2} = \frac{2\sin\phi/2\cos\phi/2}{2\sin\phi/2} = \cos\phi/2$$

$$\beta = \phi/2$$

$$\cos\alpha = \cos\phi\,\frac{\sin\phi}{\sqrt{2-2\cos\phi}} + \sin\phi\,\frac{1-\cos\phi}{\sqrt{2-2\cos\phi}}$$

$$= \frac{\sin\phi}{\sqrt{2-2\cos\phi}}.$$

$$mg\,\cancel{\sin\phi} = \frac{q^2}{\left[R\sqrt{2-2\cos\phi}\right]^2}\,\frac{\cancel{\sin\phi}}{\sqrt{2-2\cos\phi}}$$

$$\sqrt{2-2\cos\phi} = 2\sin\phi/2$$

$$mg = \frac{q^2}{R^2}\,\frac{1}{(2\sin\frac{\phi}{2})^3}$$

Equilibrium at ϕ_0 where $\boxed{\sin\frac{\phi_0}{2} = \sqrt[3]{q^2/8mgR^2}}$

$$T = \tfrac{1}{2}mR^2\dot\phi^2$$

$$V = V(\phi)$$

consider small oscillations within the angle α of ϕ_0

$$\phi = \phi_0 + \alpha$$

$$\mathcal{L} = T - V = \tfrac{1}{2}mR^2\dot\phi^2 - V(\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\dot\phi} = mR^2\dot\phi \qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} = mR^2\ddot\phi$$

#2-3

$$\frac{\partial L}{\partial \varphi} = -\frac{\partial V}{\partial \varphi} = -V'(\varphi)$$

$$\varphi = \varphi_0 + \alpha$$
$$\ddot{\varphi} = \ddot{\alpha}$$

Expand V' in Taylor series about φ_0

$$V'(\varphi) = V'(\varphi_0) + \alpha\, V''(\varphi_0)$$

$$mR^2 \ddot{\alpha} = -V'(\varphi_0) - \alpha\, V''(\varphi_0)$$

The oscillatory term is then

$$mR^2 \ddot{\alpha} = -\alpha V''(\varphi_0) \quad ; \quad \omega^2 = \frac{1}{mR^2} V''(\varphi_0)$$

$$V(\varphi) = mgR(1-\cos\varphi) + \frac{q^2}{x} = mgR(1-\cos\varphi) + \frac{q^2}{2R\sin\varphi/2}$$

$$V'(\varphi) = mgR\sin\varphi - \frac{q^2}{2R\sin^2\varphi/2}\left(\frac{1}{2}\cos\frac{\varphi}{2}\right)$$

$$V''(\varphi) = mgR\cos\varphi - \frac{q^2}{4R}\,\frac{\sin^2\frac{\varphi}{2}\left(-\frac{1}{2}\sin\frac{\varphi}{2}\right) - \cos\frac{\varphi}{2}\left(\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\right)}{\sin^4\varphi/2}$$

#2-14

$$V''(\varphi_0) = mgR\left(1-2\sin^2\frac{\varphi_0}{2}\right) - \frac{q^2}{4R}\,\frac{-\frac{1}{2}\sin^2\frac{\varphi_0}{2} - 1 + \sin^2\frac{\varphi_0}{2}}{\sin^3\frac{\varphi_0}{2}}$$

$$= mgR\left[1-2x_0^2 - \cancel{\frac{q^2}{4mgR^2}}^{\,2}\;\frac{-1+x_0^2/2}{\cancel{x_0^3}}\right]$$

$$= mgR\left[3-3x_0^2\right]$$

$$\boxed{\omega^2 = \frac{3g}{R}\left(1-x_0^2\right)}$$

#2-1 Ans 1.

(a) $dI(v) = n\, v \cos\theta\, A\, d\Omega/4\pi$

where dI is the no of escaping molecules/sec having v into the solid angle $d\Omega$ located at an angle θ from the normal to the area A of the hole. n is the number density of the gas in the container

This integrates to $I_0 = \frac{1}{4} n \bar{v} A$ where $\bar{v}$ is the mean molecular speed

b) It can be seen therefore that the velocity distribution of the escaping gas is

$\propto B\, v^3 e^{-\frac{mv^2}{2kT}}$ which can be normalized to

$$I(v)\,dv = \frac{2 I_0}{\alpha^4} v^3 e^{-v^2/\alpha^2}\, dv$$

where $I(v)$ is the total intensity of escaping gas in the velocity range v to $v + dv$ and I_0 is the total no per second escaping the container

$$\alpha^2 \equiv \frac{2kT}{m}$$

#3-2

(c) $A\int_0^\infty v^2 e^{-v^2/\alpha^2}\,dv = N_0$ (total no. of molecules in the container)

$$= A\frac{\alpha^2}{4}\sqrt{\pi\alpha^2}$$

$$\therefore A = \frac{4N_0}{\sqrt{\pi}\,\alpha^3}$$

$$N(v)\,dv = \frac{4N_0}{\sqrt{\pi}\,\alpha^3}\,v^2 e^{-v^2/\alpha^2}\,dv$$

$$\bar{v} = \frac{1}{N_0}\,\frac{4N_0}{\sqrt{\pi}\,\alpha^3}\int_0^\infty v^3 e^{-v^2/\alpha^2}\,dv = \frac{2}{\sqrt{\pi}}\alpha = \sqrt{\frac{8kT}{\pi m}}$$

$$v_{rms} = \sqrt{\frac{1}{N_0}\overline{v^2}} = \sqrt{\frac{4}{\alpha^3\sqrt{\pi}}\int_0^\infty v^4 e^{-v^2/\alpha^2}\,dv} = \sqrt{3/2}\,\alpha = \sqrt{\frac{3kT}{m}}$$

$$\left.\frac{dN(v)}{dv}\right|_{v=v_{max}} = 0 = \frac{4N_0}{\sqrt{\pi}\,\alpha^3}\left(2v_m e^{-v_m^2/\alpha^2} - v_m^2\,\frac{2v_m}{\alpha^2}\,e^{-v_m^2/\alpha^2}\right)$$

$$v_m^2 = \alpha^2 \quad ; \; v_m = \sqrt{\frac{2kT}{m}}$$

(d) In escaping beam

$$\bar{v} = \frac{2I_0}{I_0\alpha^4}\int_0^\infty v^4 e^{-v^2/\alpha^2}\,dv = \frac{3}{4}\sqrt{\pi}\,\alpha = \sqrt{\frac{9}{8}\pi\frac{kT}{m}}$$

$$v_{rms} = \sqrt{\frac{2}{\alpha^4}\int_0^\infty v^5 e^{-v^2/\alpha^2}\,dv} = \sqrt{2}\,\alpha = 2\sqrt{\frac{kT}{m}}$$

$$\left.\frac{dI(v)}{dv}\right|_{v=v_m} = 0 \qquad v_m^2 = 3/2\,\alpha^2 \quad ; \; v_m = \sqrt{\frac{3kT}{m}}$$

(e) $\langle KE\rangle = \frac{1}{2}m\left(v_{rms}^{beam}\right)^2 = \frac{1}{2}m\,\frac{4kT}{m} = 2kT$

(f) Various means of measuring velocity in escaping beam

i. Chopper, deflection by gravity or elec/mag fields

#4-1

E&M (a) consider fields as $E(\vec r,t) = E(\vec r)e^{-i\omega t}$ etc. in a source free region.

M.E. $\nabla \times \vec E(\vec r) = -i\frac{\omega}{c}\vec B(\vec r)$

$\nabla \cdot \vec E(\vec r) = 0$

$\nabla \times \vec B(\vec r) = +i\mu\epsilon\frac{\omega}{c}\vec E(\vec r)$

$\nabla \cdot \vec B(\vec r) = 0$

Propagation in the z direction where

$\therefore \quad E(\vec r) = \vec E_t(x,y)\,e^{-i\beta z} \quad (\hat k\beta = \vec k)$

$B(\vec r) = \vec B_t(x,y)\,e^{-i\beta z}$

write $\nabla = \{\hat i\,\partial/\partial x + \hat j\,\partial/\partial y\} + \hat k\frac{\partial}{\partial z} = \nabla_t + \hat k\,\partial/\partial z$

$\therefore \quad \nabla\times\vec E(\vec r) = (\nabla_t\times\vec E_t)e^{-i\beta z} - i\beta(\hat k\times\vec E_t)e^{-i\beta z}$

$\nabla\cdot\vec E(\vec r) = (\nabla_t\cdot\vec E_t)e^{-i\beta z}$

similarly for B.

put in M.E. & drop $e^{-i\beta z}$ factor

$\vec\nabla_t\times\vec E_t - i\beta(\hat k\times\vec E_t) = -\frac{i\omega}{c}\vec B_t$

$\nabla_t\cdot\vec E_t = 0$

$\nabla_t\times\vec B_t - i\beta(\hat k\times B_t) = i\,\omega/c\;\epsilon\mu\,\vec E_t$

$\nabla_t\cdot\vec B_t = 0$

and $\nabla_t\times\vec E_t(\vec B_t)$ have only z components

$\hat k\times\vec E_t(\vec B_t)$ $\perp$ to $\hat k$ so only transverse components.

$\therefore$ (1) $\nabla_t\times\vec E_t = 0$ (1) $\nabla_t\times\vec B_t = 0$

(2) $\nabla_t\cdot\vec E_t = 0$ (2) $\nabla_t\cdot\vec B_t = 0$

(3) $\beta\,\hat k\times\vec E_t = \omega/c\,\vec B_t$ (3) $\beta\,\hat k\times\vec B_t = -\omega/c\,\mu\epsilon\,\vec E_t$

$\vec E_t$ defined by (1) & (2) as in electrostatics

similarly $\vec B_t$ by ...   Q.E.D. part (a)

Ans (2)

#4-2

$\hat{n} \times (3)$ & insert (6) a result

$$\beta\, \hat{k}\times(\hat{k}\times E_t) = \omega/c\,(\hat{k}\times\vec{B}_t) = \frac{\omega}{c}\frac{-1}{\beta}\frac{\omega}{c}\,\mu\epsilon\,\vec{E}_t$$

& $\hat{k}\times(\hat{k}\times E) = -\vec{E}_t$

$$\therefore \quad \vec{E}_t = \frac{1}{\beta^2}\frac{\omega^2}{c^2}\mu\epsilon\,\vec{E}_t \quad \text{or} \quad \beta^2 = \frac{\omega^2\mu\epsilon}{c^2}$$

$$\therefore \quad \vec{E}_t = -\frac{1}{\beta}\frac{\omega}{c}\,\hat{k}\times\vec{B}_t = -\frac{1}{\sqrt{\mu\epsilon}}\,h\times\vec{B}_t$$

or equiv. $\vec{B}_t = \sqrt{\mu\epsilon}\,\hat{k}\times\vec{E}_t$ QED part (b)

1) Paradox because wave were taken to flow in one direction only. If the line is terminated in an arbitrary impedance wave will flow in both directions so we must consider term $\sim e^{+i\beta z}$

$$\therefore \quad \vec{E}_t(\vec{r}) = E_{tR}\,e^{-i\beta z} + E_{tL}\,e^{+i\beta z} \quad \text{and}$$

$$\vec{B}_t\,\vec{r} = \sqrt{\mu\epsilon}\left[\vec{B}_{tR}\,e^{-i\beta z} - \vec{B}_{tL}\,e^{+i\beta z}\right]$$

E_{tR} & E_{tL} are arbitrary so $|\vec{E}_t(\vec{r})|/|\vec{B}_t(\vec{r})|$ is not fixed but determined by details of the terminating or load impedance.

$$A + B \times c = ?$$

#5-1

(a) Avg. occupation number is, for ideal Bose gas,

$$\langle n \rangle = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

If, say the smallest value of ϵ is 0 then, unless $\mu \leq 0$ $\langle n \rangle \leq 0$ which is physically impossible.

(b) As $T \to 0$ all particles are in lowest energy state $\epsilon = 0$, so, since macroscopically $N >> 1$

$$N = \frac{1}{e^{-\beta\mu}-1} \qquad \beta = \frac{1}{k_B T}$$

$$e^{-\beta\mu}-1 = \frac{1}{N}$$

$$e^{-\beta\mu} = 1+\frac{1}{N}$$

$$e^{\beta\mu} = \frac{1}{1+\frac{1}{N}} \cong 1 - \frac{1}{N} \qquad (N >> 1)$$

(c) These "particles" are not number conserved. They are really just descriptions of energy excitations so for them $\mu \equiv 0$.

(d) Since T_c is the smallest temperature for which there are still N particles in excited states

$$N = \int_0^\infty \frac{1}{e^{\beta(\epsilon-\mu)}-1} \, \mathcal{D}(\epsilon)\, d\epsilon$$

$\mathcal{D}(\epsilon)$ is the density of states.

#5-2

We seek the minimum value of T, i.e. T_c and $\mu(T_c)$ which still permit

$$N = \int_0^\infty \frac{1}{e^{\beta_c(\epsilon - \mu_c)} - 1} \, d\epsilon \, \mathcal{D}(\epsilon)$$

Since T_c is ~ 0 $\mu_c \cong 0$

and

$$N = \int_0^\infty \frac{1}{e^{\beta_c(\epsilon)} - 1} \, \mathcal{D}(\epsilon) \, d\epsilon$$

determines β_c.

Using the integral

$$\int_0^\infty dx \, \frac{\sqrt{x}}{e^x - 1} = 1.3\sqrt{\pi}$$

$$N = \frac{C}{\beta_c^{3/2}} \, 1.3\sqrt{\pi}$$

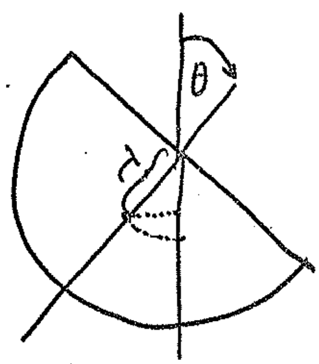
$$\frac{1}{\beta_c} = \left(\frac{N}{V} \, \frac{1}{1.3\, C \sqrt{\pi}} \right)^{2/3}$$

#6-1

(a) $L = T - V \qquad V = Mg\lambda(1-\cos\theta)$

$$T = T_{cm} + T_{\text{around cm}}$$
$$= T_{cm} + \frac{1}{2} I \dot{\theta}^2$$

$$T_{cm} = \frac{1}{2} M (\dot{x}_{cm}^2 + \dot{y}_{cm}^2)$$



Since there is no friction

$\dot{x}_{cm} = 0$
(no horizontal motion)

$$\dot{y}_{cm} = \frac{d}{dt}$$
$$= \frac{d}{dt}(\lambda - \lambda\cos\theta)$$
$$= \dot{\theta}\lambda\sin\theta$$

$$L = \frac{1}{2} M (\dot{\theta}^2 \lambda^2 \sin^2\theta) + \frac{1}{2} I \dot{\theta}^2 - Mg\lambda(1-\cos\theta)$$

(b) In this case $\dot{X}_{cm} \neq 0$ and is coupled to the rolling (rocking) by

$$\dot{X}_{cm} = \frac{d}{dt}(R\theta - \lambda\sin\theta)$$
$$= R\dot{\theta} - \lambda\dot{\theta}\cos\theta$$

#6 ~

Thus $\mathcal{L} = \frac{1}{2} M \left(\dot{\theta}^2 d^2 \sin^2\theta + R^2\dot{\theta}^2 + d^2\dot{\theta}^2\cos^2\theta \right.$

$\left. - 2Rd\,\dot{\theta}^2 \cos\theta \right) + \frac{1}{2} I \dot{\theta}$

$- Mgd(1 - \cos\theta)$

$= \frac{1}{2} M \left(d^2\dot{\theta}^2 + R^2\dot{\theta}^2 - 2Rd\cos\theta \;\; \dot{\theta}^2 \right) + \frac{1}{2} I \dot{\theta}^2$

$- Mgd(1 - \cos\theta)$

(C) Case (a) $\quad \sin^2\theta \sim \theta^2$

$\mathcal{L} = \frac{1}{2} M \left(\dot{\theta}^2 d^2 \theta^2 \right) + \frac{1}{2} I \dot{\theta}^2 - Mgd\left(\frac{\theta^2}{2}\right)$

Which becomes since $\dot{\theta} \sim A \quad \theta \sim A$

$\mathcal{L} = \frac{1}{2} I \dot{\theta}^2 - \frac{Mgd}{2}\,\theta^2$

To find Eq. of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$I\ddot{\theta} + Mgd\,\theta = 0$$

$$\omega^2 = \frac{Mgd}{I}$$