This Comprehensive Examination for Winter 1975 (#20) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook — you may get some credit for it!

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books and papers, except pencil and bluebook, on the floor.

Some information you may find useful:

$$\pi = 3.14159$$

$$h \cong 6.63 \times 10^{-34} \text{ joule-sec.}$$

$$(1+x)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} x^i$$

$$(1+\delta)^n \approx 1 + n\delta + \ldots \quad \delta << 1$$

$$\rho e^{i\phi} = x + iy$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mu_\beta = 9 \times 10^{-21} \text{ erg/gauss} = 9 \times 10^{-24} \text{ J/T}$$

$$\ln N! \cong N \ln N - N$$

If you need a table of integrals, ask the proctor.

1. A bead of mass m is constrained to move on a frictionless parabolic wire having the equation

$$z = A x^2$$

with the plane of the wire normal to the earth's surface.

(4 pts) a) What is meant by a holonomic constraint in classical mechanics? Give an example of a non-holonomic constraint.

(8 pts) b) Use a Lagrangian method to determine the equations of motion for the constrained bead.

(8 pts) c) Find the frequency of small-amplitude oscillations executed by the bead.

2. A particle with mass m is confined to a one-dimensional infinite square well of dimension <u>a</u>:

$$-\frac{a}{2} \leq x \leq \frac{a}{2} .$$

The normalized wave function at time t=0 is given by

$$\psi(x,0) = \sqrt{\frac{8}{5a}} \left(\cos \frac{\pi x}{a} + \frac{1}{2} \sin \frac{2\pi x}{a}\right) .$$

(10 pts) a) Calculate $\psi(x,t)$ for $t > 0$.

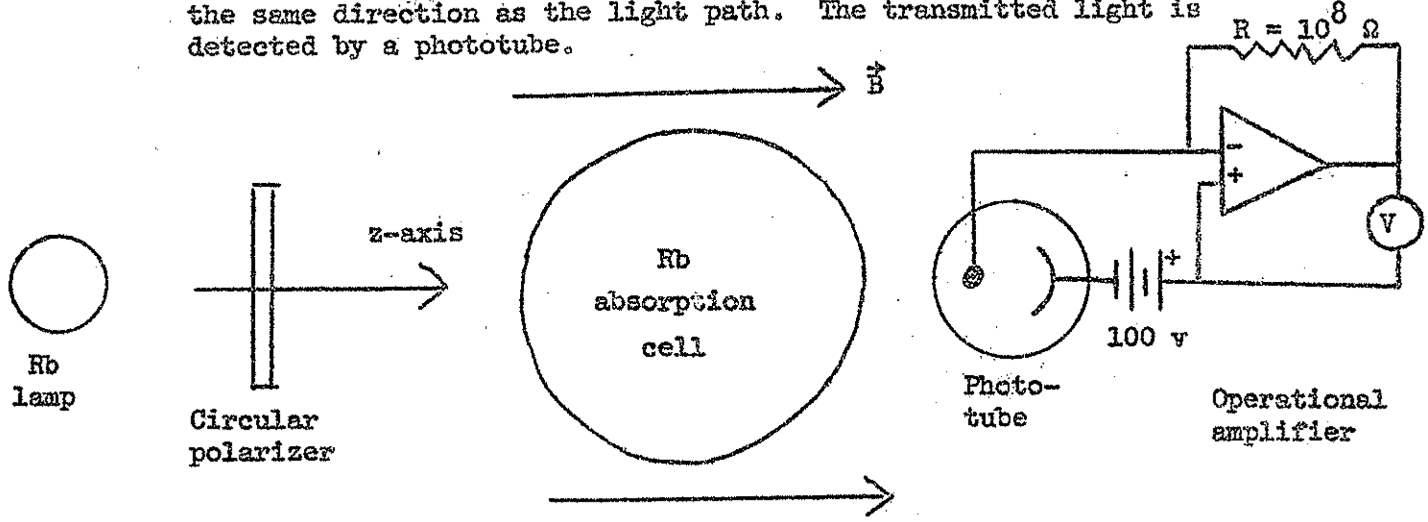
(10 pts) b) Find the probability current as a function of time.

3. Calculate the minimum laboratory kinetic energy necessary to produce an antiproton from a proton-proton collision.

$$p + p \rightarrow p + p + p + \bar{p}$$

Assume that one of the initial protons is at rest in the laboratory reference frame.

4. In an optical pumping experiment, 7900 Å resonance radiation from a Rb^{87} discharge lamp is passed through a circular polarizer and then through a glass-walled absorption cell containing Rb^{87} vapor at 40°C. A uniform static magnetic field $\vec{B}$ is applied to the absorption cell, in the same direction as the light path. The transmitted light is detected by a phototube.



In a simplified analysis of the experiment we may treat the atom as having a $^2P_{1/2}$ excited state and a $^2S_{1/2}$ ground state, neglecting the effect of all other states. Rb^{87} has a nuclear spin $I = \frac{3}{2}$, which gives rise to hyperfine splitting of the levels.

(5 pts) a) Draw an energy level diagram showing the $^2S_{1/2}$ hyperfine components and their Zeeman shifts due to the magnetic field $\vec{B}$. Consider only the weak-field limit, and label all states using the appropriate quantum numbers. (No calculation is necessary.)

(5 pts) b) If the circular polarizer passes only left-hand circularly polarized light (photons having positive helicity), list the quantum numbers for all allowed electronic transitions by which excitation can take place in the absorption cell. Assume that the Doppler width of the lamp light is greater than the energy spread of the Zeeman levels.

(5 pts) c) If B = 20 gauss, what will be the frequency of a transition between adjacent ground-state Zeeman levels? The magnitude of the atomic g-factor is $\frac{1}{2}$ for all states. Explain what happens to the photocell signal when a transverse oscillatory magnetic field at this frequency is applied to the atoms in the absorption cell.

(5 pts) d) The voltmeter in the operational amplifier circuit reads 0.1 volts, and the phototube has a quantum efficiency of 1%. Calculate the power in the light beam striking the photocathode.

5. Consider a linear array of N identical equally spaced atoms of spin 1/2. Assume the only interaction to be between nearest neighbors and to be of the form $J\sigma_z^{i}\sigma_z^{j}$ for the ij^th pair; that is, the interaction energy is J if i and j are parallel and -J if i and j are antiparallel. In the limit of vanishing applied field, the specific heat at constant field may be shown to be equivalent to the specific heat at constant magnetization, which is in turn equal to the specific heat at constant J. Find an expression for the heat capacity per atom at constant magnetization for such a system in zero applied field.

[Hint: It may be useful to assume that $C_M = \left(\frac{\partial U}{\partial T}\right)_J$ or $C_M = T\left(\frac{\partial^2 F}{\partial T^2}\right)_J$.]

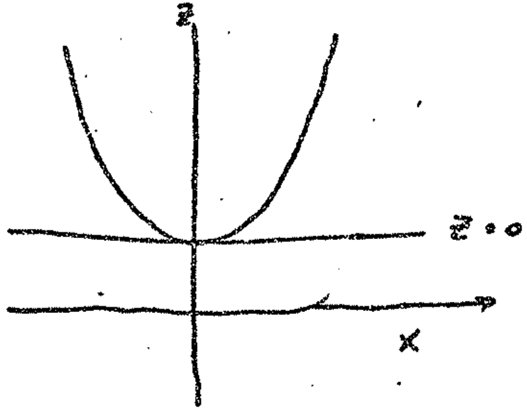
6. Make an order-of-magnitude estimate of the lifetime of a free hydrogen atom in the $2\ {}^2S_{1/2}$ state which moves with a velocity $v = 10^5$ cm/sec perpendicular to a homogeneous constant magnetic field $B = 10^3$ gauss. The field-free 2S lifetime is $\tau_{2S} = 10^{-1}$ sec, the Lamb-shift frequency $(2P_{1/2} - 2S_{1/2})$ is $\nu_L = 10^3$ MHz, and the 2P lifetime is $\tau_{2P} = 10^{-9}$ sec.

1. (a) A holonomic constraint relates the coordinates and time with a functional equality. This functional must be differentiable with respect to the coordinates.

A non-holonomic constraint is one that does not relate the coordinates by an equality, but may do so by an inequality of the form, e.g.

$$\phi(\bar{x}_1, \bar{x}_2, \bar{x}_3 \cdots \bar{x}_N, t) \geq 0$$

(b) The Lagrangian may be written, using the coordinates in the diagram. (The problem can be worked in parabolic coordinates, but this goes beyond present requirements)



$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{z}^2) - mgz$$

The constraint is holonomic and is

$$Ax^2 - z = 0 .$$

We can include the constraint directly, or we can impose it by the method of Lagrange multipliers. The latter method will be demonstrated.

Thus, with the Lagrange Multiplier, λ,

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{z}^2) - mgz + \lambda (Ax^2 - z)$$

Finding the equations of motion proceeds by the Euler - Lagrange conditions

$$(I, II) \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \qquad i = 1, 2$$

$$(III) \quad Ax^2 - z = 0$$

$$\text{I.} \quad m\ddot{x} - 2A\lambda x = 0$$

$$\text{II.} \quad m\ddot{z} + mg + \lambda = 0$$

$$\text{III} \quad Ax^2 - z = 0$$

(c) What is required now is to time differentiate (III), thus

$$\frac{d}{dt}(Ax^2 - z) = 0 = 2Ax\dot{x} - \dot{z}$$

$$\frac{d}{dt}(2Ax\dot{x} - \dot{z}) = 0 = 2A(\dot{x}^2 + x\ddot{x}) - (\ddot{z})$$

So that eq. (II) becomes

$$2Am(\dot{x}^2 + x\ddot{x}) + mg + \lambda = 0$$

We can now replace λ in (I.)

$$m\ddot{x} + 2Ax\left[(2Am)(\dot{x}^2 + x\ddot{x}) + mg\right] = 0$$

or

$$m\ddot{x} + 2Amgx + 4A^2mx(\dot{x}^2 + x\ddot{x}) = 0$$

The last term is obviously proportional to the third power of the amplitude, while the first two are proportional to the first power. So that in the limit of small amplitudes we have

$$m\ddot{x} + 2Amgx = 0$$

$$\omega^2 = 2Ag$$

2.

(a) At time $t=0$

$$\psi(x,0) = \sqrt{\frac{8}{5a}}\left(\cos\frac{\pi x}{a} + \frac{1}{2}\sin\frac{2\pi x}{a}\right)$$

For later times we can write

$$\psi(x,t) = \sum_n C_n(t)\,\phi_n(x)$$

where

$$C_n(t) = \left[\int \phi_n^*(x')\,\psi(x',0)\,dx'\right] e^{-\frac{i}{\hbar}E_n t}$$

and

$$-\frac{\hbar^2}{2m}\frac{\delta}{\partial x^2}\phi_n(x) = E_n\phi_n(x) \qquad -\frac{a}{2} < x \leq \frac{a}{2}$$

The eigenvalue problem is the familiar one dimensional box with infinite walls containing a single particle of mass m.

These properly normalized solutions are

$$\left\{\begin{array}{ll} \sqrt{\frac{2}{a}}\cos\frac{n\pi}{a}x & n \text{ odd} \\ & \\ \sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x & n \text{ even} \end{array}\right\} \quad E_n = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2}$$

Thus there are only 2 coef.

$$c_1(t) = \frac{a}{2}\sqrt{\frac{8}{5a}}\sqrt{\frac{2}{a}}\,e^{-\frac{i}{\hbar}E_1 t} \qquad c_2(t) = \frac{a}{2}\cdot\frac{1}{2}\cdot\sqrt{\frac{8}{5a}}\sqrt{\frac{2}{a}}\,e^{-\frac{i}{\hbar}E_2}$$

2 (cont'd)

So

$$\Psi(x,t) = \sqrt{\frac{8}{5a}} \left(e^{-\frac{i}{\hbar}E_1 t} \cos\frac{\pi x}{a} + \frac{1}{2} e^{-\frac{i}{\hbar}E_2 t} \sin\frac{2\pi x}{a} \right)$$

(b) The Probability current is (in one dim)

$$J_x = \frac{\hbar}{2mi}\left[\Psi^* \frac{\partial}{\partial x}\Psi - \Psi \frac{\partial}{\partial x}\Psi^*\right]$$

$$J_x = \left(\frac{8}{5a}\right)\frac{\hbar}{m}\left[\phi_1(x)\,\tilde{\phi}_2'(x)\,\sin(\omega_1-\omega_2)t - \phi_1'(x)\,\tilde{\phi}_2(x)\,\sin(\omega_1-\omega_2)\right]$$

where $\phi_1(x) = \cos\frac{\pi x}{a}$ $\qquad \phi_1'(x) = \frac{\pi}{a}\sin\frac{\pi x}{a}$

$\phi_2(x) = \frac{1}{2}\sin\frac{2\pi x}{a}$ $\qquad \phi_2'(x) = -\frac{1}{2}\times\frac{2\pi}{a}\cos\frac{2\pi x}{a}$

$$= -\frac{\pi}{a}\cos\frac{2\pi x}{a}$$

$\omega_i = \frac{E_i}{\hbar}$

3.

Relativity Solution

(a) $p + p \rightarrow p + p + p + \overline{p}$

in C of M system, energy of final products at threshold = rest energy only. Energies of incident and target protons are equal

$$\therefore 2mc^2 = 4m_0c^2$$

$$\frac{m}{m_0} = 2 = \frac{1}{\sqrt{1-\beta_{cm}^2}}$$

$$\beta_{cm}^2 = 3/4$$

to transform back to lab system

$$v_{Lab} = \frac{2v_{cm}}{1+\frac{v_{cm}^2}{c^2}}$$

$$\beta_{Lab} = \frac{2\beta_{cm}}{1+\beta_{cm}^2}$$

$$\sqrt{1-\beta_{Lab}^2} = \frac{1-\beta_{cm}^2}{1+\beta_{cm}^2} = \frac{1}{7}$$

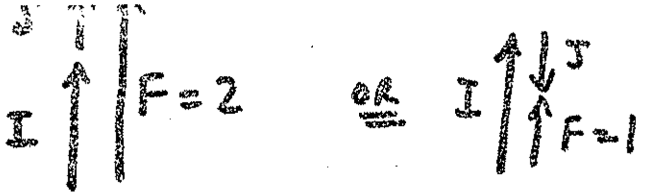
$$\therefore E_{Lab} = \frac{m_0c^2}{1/7} = 7m_0c^2$$

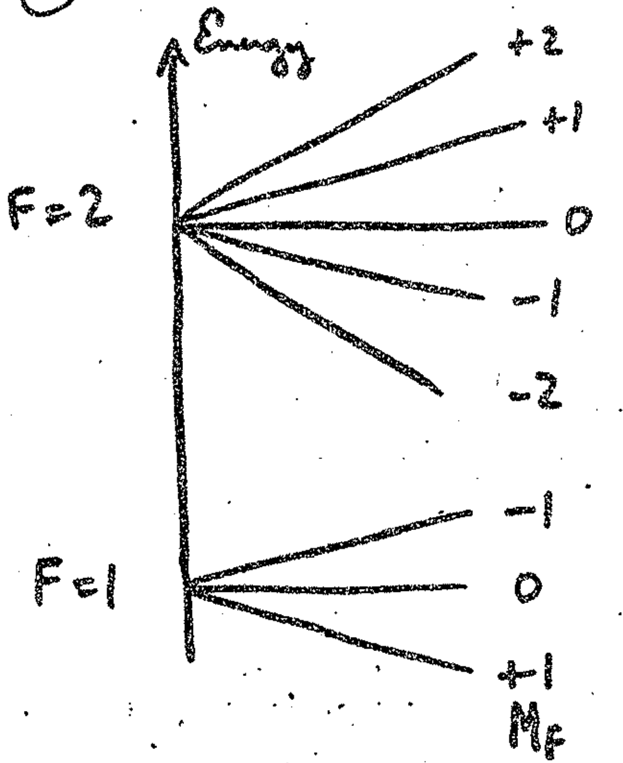
$$\text{Kinetic Energy} = 7m_0c^2 - m_0c^2 = 6m_0c^2$$

For proton $m_0 = 938\,\text{MeV}$, $\therefore$ $\underline{K.E. = 5.6\ \text{BeV}}$

4. [illegible]

ⓐ $J=\frac{1}{2}$, $I=\frac{3}{2}$





Note that the F=1 Zeeman levels are inverted, since the magnetic moment (antiparallel to $\vec{J}$) is now parallel to $\vec{F}$. $g_{F=1}$ is negative

ⓑ Selection rules : $\Delta F = 0, \pm 1$; $\Delta M_F = +1$.

$^2S_{1/2}$		$^2P_{1/2}$
(F = 2) $M_F = -2$	$\longrightarrow$	(F = 2) $M_F = -1$
(F=1, 2) $M_F = -1$	$\longrightarrow$	(F=1, 2) $M_F = 0$
" 0	$\longrightarrow$	" +1
" +1	$\longrightarrow$	" +2
(F=2) +2 : No transition is possible.		

If there were no depolarizing collisions, all atoms would be "pumped" into the F=2, $M_F = 2$ ground-state sublevel. The absorption cell would be [illegible] — Bohr magneton

Ⓒ $h\nu = \Delta E = g_F \mu_B B$

$\nu = g_F \frac{\mu_B}{h} B = \frac{1}{2} \times 1.4\,\text{MHz/gauss} \times 20\,\text{gauss}$

$= \underline{\underline{14\ \text{MHz}}}$. (radio frequency).
(at resonance)

When the rf field is turned on, transitions are induced between adjacent Zeeman levels, mixing the states. The populations of the states become more nearly equal, and therefore there are more Rb atoms capable of absorbing the resonance radiation. Consequently the photocell signal <u>decreases</u> when the rf field is switched on.

~~ⓓ $\Delta E^{(2)} \approx \frac{(\mu_B B)^2}{h\nu_{HF}}$ using perturbation theory, or by dimensional analysis. (This also follows from diagonalization of the Hamiltonian matrix.)~~

~~$\therefore \Delta\nu^{(2)} \approx \left(\frac{\mu_B}{h}\right)^2 \frac{B^2}{\nu_{HF}}$~~

~~$= \left(1.4 \frac{\text{MHz}}{\text{gauss}}\right)^2 \times \frac{(20\,\text{gauss})^2}{6.8 \times 10^3\,\text{MHz}} = \frac{2 \times 400}{7000}\,\text{MHz} \approx \underline{\underline{0.1\,\text{M}}}$~~

ⓐ $V = iR$ for operational amplifier
(V = output voltage ; i = input current ; R = feedback resisto

$Q \equiv \frac{N_e}{N_\gamma}$ Quantum efficiency.

$P = h\nu_R \frac{dN_\gamma}{dt}$ Light po
↖ Resonance freq
from lamp.

$\frac{V}{R} = i = e\frac{dN_e}{dt} = eQ\frac{dN_\gamma}{dt}$

$P = h\nu_0 \cdot \frac{V}{RQ} = (1\,\text{eV}) \cdot \frac{0.1\,V}{[illegible]} = 10^{-7}$ watt

5.

Statistical Mechanics (Graduate) Solution
"Of Thee Ising"

There are $N-1$ pairs ; suppose N_p are parallel & N_a antiparallel

$$N_p + N_a = N-1$$

Energy for given configuration $E = J(N_p) - J(N_a)$

$$E = J(2N_p - N + 1)$$

partition function $Z = \sum e^{-E/kT}$

$$Z = \boxed{2} \sum_{N_p=0}^{N-1} \boxed{\frac{(N-1)!}{N_a!\,N_p!}} \exp\left[-\frac{J(2N_p-N+1)}{kT}\right]$$

flipping all spins doesn't change energy

identical particles, number of identical ways to have N_p & N_a

$$= 2\exp\left[\frac{J(N-1)}{kT}\right] \sum_{N_p=0}^{N-1} \frac{(N-1)!}{(N-1-N_p)!\,N_p!} e^{-2JN_p/kT}$$

$$\boxed{(1+x)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} x^i}$$

$$\therefore Z = 2\exp\left[\frac{J(N-1)}{kT}\right] \left[1 + e^{-2J/kT}\right]^{N-1}$$

$$Z = 2\left[2^{N-1}\right] \left[\frac{e^{J/kT} + e^{-J/kT}}{2}\right]^{N-1}$$

$$\boxed{Z = 2^N \left[\cosh J/kT\right]^{N-1}}$$

(given on cover sheet)

5. (cont'd) other needed formulae

$\operatorname{sech} u = \frac{1}{\cosh u}$

$d(\cosh u) = \sinh u \, du$ $\quad \cosh u = (e^{u}+e^{-u})/2$

$d(\sinh u) = \cosh u \, du$ $\quad \sinh u = (e^{u}-e^{-u})/2$

$d(\tanh u) = \operatorname{sech}^2 u \, du$ $\quad \tanh u = \sinh u/\cosh u$

$$F = -kT \ln Z = -kT\left[N \ln 2 + (N-1)\ln \cosh \frac{J}{kT}\right]$$

$$S = -\frac{\partial F}{\partial T} = kT\left[(N-1)\frac{1}{\cosh\frac{J}{kT}}\left(\sinh\frac{J}{kT}\right)\left(-\frac{J}{kT^2}\right)\right] + k\left[N\ln 2 + (N-1)\ln\cosh\right.$$

$C_M = C_J$

$$C_M = T\frac{\partial S}{\partial T} = kT\left[(N-1)\left(-\frac{J}{kT^2}\right)\tanh\frac{J}{kT}\right] + kT^2\left[(N-1)\left(\frac{2J}{kT^3}\right)\tanh\frac{J}{kT}\right]$$

$$+\cancel{k}T^2\left[(N-1)\left(-\frac{J}{\cancel{kT^2}}\right)\operatorname{sech}^2\frac{J}{kT}\left(-\frac{J}{kT^4}\right)\right]$$

$$+kT(N-1)\frac{1}{\cosh\frac{J}{kT}}\sinh\frac{J}{kT}\left(-\frac{J}{kT^2}\right)$$

$$C_H = (N-1)\frac{J^2}{kT^2}\operatorname{sech}^2\frac{J}{kT}$$

$$\frac{C_H}{N} \approx \frac{C_H}{N-1} = \boxed{\frac{J^2}{kT^2}\operatorname{sech}^2\frac{J}{kT}}$$

6.

Solution: The motional electric field is $\vec{E} = \frac{\vec{v}}{c}\times\vec{B}$.

$\vec{E}$ couples $2S_{1/2}$ to $2P_{1/2}$: $\Psi = \Psi_{2S} + \epsilon\,\Psi_{2P}$

where $\epsilon = \frac{\langle 2P|eEz|2S\rangle}{h\nu_L}$

The $2P_{1/2}$ part of Ψ decays by photon emission:

$$\Gamma = |\epsilon|^2\,\Gamma_{2P} + \Gamma_{2S} \qquad \epsilon \approx \frac{e v B a_0}{h\nu_L c}$$

$$\epsilon \approx \frac{5\times10^{-10}\times10^{5}\times10^{3}\times5\times10^{-9}}{7\times10^{-27}\times10^{9}\times3\times10^{10}} = \frac{10^{-11}}{10^{-8}} = 10^{-3}$$

$$\frac{1}{\tau} = \Gamma = 10^{-6}\times10^{+9}\,\text{sec}^{-1} + \cancel{10\,\text{sec}^{-1}} \overset{\text{DROP}}{} \longrightarrow \underline{\underline{\tau = 10^{-3}\,\text{sec}}}$$

3. (cont'd.)

$$C_N = (N-1)\left(\frac{4J^2}{kT^2}\right)\left[\frac{e^{-J/kT}}{1+e^{-2J/kT}}\right]^2$$

$$= (N-1)\,\frac{J^2}{kT^2}\left[\frac{2}{e^{J/kT}+e^{-J/kT}}\right]^2 = (N-1)\,\frac{J^2}{kT^2}\,\operatorname{sech}^2 J/kT$$

$$\frac{C_N}{N} \approx \frac{C_N}{N-1} = \underline{\underline{\frac{J^2}{kT^2}\,\operatorname{sech}^2 J/kT}}$$

5. (cont'd)

Alternate Solution

$$U = \frac{\displaystyle\sum_{N_p=0}^{N-1} \frac{(N-1)!}{(N-1-N_p)!\,N_p!}\, e^{-J(2N_p-N+1)/kT}\, J(2N_p-N+1)}{\displaystyle\sum_{N_p=0}^{N-1} \frac{(N-1)!}{(N-1-N_p)!\,N_p!}\, e^{-J(2N_p-N+1)/kT}}$$

$$= \frac{2J \displaystyle\sum_{N_p=0}^{N-1} \overset{\boxed{(N-1)!/(N-1-N_p)!\,N_p!}}{N_p}\, e^{-J(2N_p/kT)}}{\displaystyle\sum_{N_p=0}^{N-1} \frac{(N-1)!}{(N-1-N_p)!\,N_p!}\, e^{-J(2N_p/kT)}} + \boxed{J(-N+1)}$$ neglect constant term

denominator $= \left(1+e^{-2J/kT}\right)^{N-1}$ as previously

numerator $= \displaystyle\sum_{\boxed{N_p=1}}^{N-1} \frac{(N-1)!}{(N-1-N_p)!\,(N_p-1)!} = \sum_{N_p'=0}^{N-2} \frac{(N-2)!}{(N-2-N_p')!\,(N_p')!}\, e^{-2J(N_p'+1)/kT}$

$$= (N-1)e^{-2J/kT} \sum_{N_p'=0}^{N-2} \frac{(N-2)!}{(N-2-N_p')!\,(N_p')!}\, e^{-2JN_p'/kT}$$

$$= (N-1)\, e^{-2J/kT} \left(1+e^{-2J/kT}\right)^{N-2}$$

$$\therefore U = \frac{2J(N-1)\,e^{-2J/kT}}{1+e^{-2J/kT}}$$

$$C_H = \left.\frac{\partial U}{\partial T}\right)_J = 2J(N-1)\,\frac{e^{-2J/kT}\left(2J/kT^2\right)}{\left(1+e^{-2J/kT}\right)^2}$$