FALL 1974

This Comprehensive Examination for Fall 1974 (#19) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half are graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook --you may get some credit for it!

If something is omitted from the statement of the problem or you feel there are ambiguities, get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books and papers, except pencil and bluebook, on the floor.

Some information you may find useful follows:

$\pi = 3.14159$

$h \cong 6.63 \times 10^{-34}$ joule-sec.

$(1+\delta)^n \cong 1+n\delta+ \ldots.\quad \delta \ll 1$

$\rho e^{i\phi} = x+iy$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mu_\beta = 9 \times 10^{-21}$ erg/gauss $= 9 \times 10^{-24}$ J/T

$\ln N! \cong N \ln N - N$

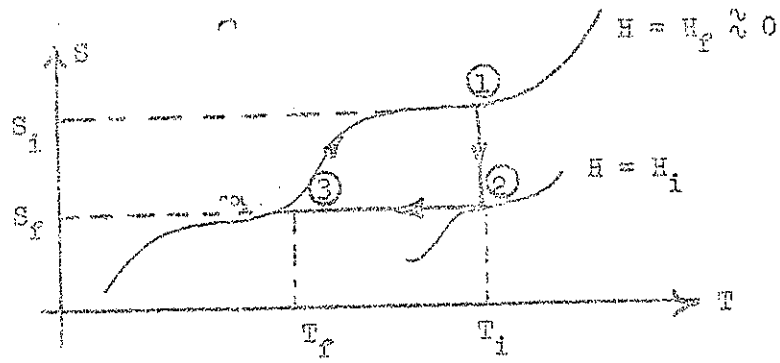
If you need a table of integrals, ask the proctor.

1. An airplane of mass $m = 10^4$ kg touches down at a speed of 180 km/hr. It has 8 wheels, each of radius $r = 1$ meter and moment of inertia $I = 100$ kg-m^2 about its own axis of rotation. If, at the instant before touchdown, the wheels are not spinning, and after touchdown the pilot uses neither engines nor brakes (and air resistance is neglected), what will be the speed of the plane after the wheels stop skidding?

2. a) A luminous object and a screen are separated by a fixed distance L. When a thin converging lens of focal length f is inserted between the object and the screen, it is observed that there are two positions of the lens for which a real image appears in focus on the screen. Calculate the distance χ between these two positions of the lens.

   b) A doubly refracting crystal has indices of refraction n_S and n_F for light polarized along the "slow" and "fast" axes, respectively. It is desired to cut a quarter wave plate from this material. The light is monochromatic and has angular frequency ω. What should be the thickness of the plate?

3. In an experiment to achieve ultralow temperatures by adiabatic demagnetization of a spin 1/2 paramagnetic salt the cycle may be represented on an entropy-temperature plot as follows:



The cycle 1 → 2 → 3 → 1 is performed by first isothermally magnetizing the sample in a field H_i, by demagnetizing it adiabatically to a field $H_f \approx 0$, and by allowing small residual heat input (due to less than perfect thermal insulation) to slowly warm the sample back to T_i.

a) Assuming an electronic g-factor $g = 2$, complete the following table for the energy splitting

$$\Delta E = E(J_Z = +1/2) - E(J_Z = -1/2)$$

and population ratio

$$R = \frac{N(J_Z = +1/2)}{N(J_Z = -1/2)}$$

when the system is in states 1, 2 and 3.

State	ΔE	R
1	≈ 0	1
2		
3		

b) Assuming the populations to be invariant in the process 2 → 3 show that

$$\frac{H_i}{T_i} = \frac{H_f}{T_f}$$

c) From statistical considerations only, calculate the change in entropy $\Delta S = S_i - S_f$ for an ensemble of N molecules. Assume only the lowest energy state to be populated at $T = T_f$ and assume all states to be energy degenerate at $H_f \approx 0$.

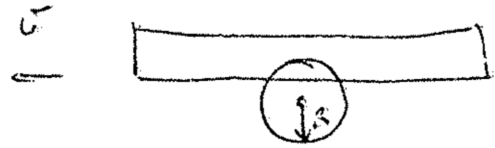
d) Based on the S-T graph given above, sketch the behavior of the heat capacity at constant field, C_H, as a function of T in the range of T as given above. Be sure to indicate the points T_i and T_f. Keeping in mind the residual heat leak, what value of the temperature would be most likely shortly after the demagnetization?

e) In practice, the field may not be reduced to zero, owing in part to the small but non-vanishing electronic dipole field. Estimate the ultimate temperature using the result of part (b) by estimating reasonable values of H_i, H_f, and T_i.

4. A popular science writer has claimed that Venus was inserted into its present orbit by ejection from Jupiter. If this event did occur, the very least one might expect is that the earth's nearly circular orbit would be sufficiently disturbed to undergo radial oscillations. What would be the period τ of these radial oscillations? Calculate τ/T, where T is the earth's orbit period.

Ph 211

1.



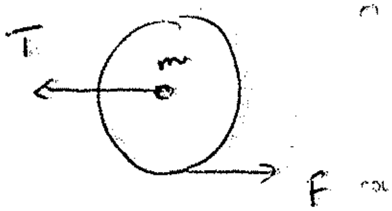
m = mass of single wheel

I = moment of inertia of single wheel

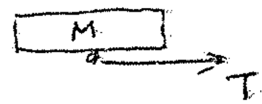
M = mass of fuselage

$\mathcal{M} = M + 8m$

Forces on wheels



Forces on Fuselage



$$-T = Ma$$

$$\left.\begin{aligned} T - F &= 8ma \\ RF &= 8I\alpha \end{aligned}\right\} \text{(8 wheels)}$$

eliminating F & T between the equations

$$-Ma - \frac{8I\alpha}{R} = 8ma$$

integrating over time

$$-M\Delta v - \frac{8I}{R}\Delta\omega = 8m\Delta v$$

But $\Delta v = v_f - v_0$

$\Delta\omega = \omega_f - \omega_0 = \omega_f = \frac{v_f}{R}$ As no slip condition!

So $$v_f = \frac{(M+8m)\, v_0}{(M+8m) + \frac{8I}{R^2}} = \frac{\mathcal{M}}{\mathcal{M} + \frac{8I}{R^2}}\, v_0$$

2. Undergrad. Optics

2.) A luminous object and a screen are separated by a fixed distance L. When a thin converging lens of focal length f is inserted between the object and the screen, it is observed that there are two positions of the lens for which a real image appears in focus on the screen. Calculate the distance x between these two positions of the lens.

$$\frac{1}{S_o} + \frac{1}{S_i} = \frac{1}{f} \qquad S_o + S_i = L$$

$$\frac{1}{S_o} + \frac{1}{L-S_o} = \frac{1}{f}$$

$$(L-\cancel{S_o})f + \cancel{S_o f} = S_o(L-S_o)$$

$$S_o^2 - L S_o + Lf = 0$$

$$S_o = \frac{1}{2}\left[L \pm \sqrt{L^2-4Lf}\right]$$

$$x = S_o^{(+)} - S_o^{(-)} = \underline{\underline{\sqrt{L^2-4Lf}}}$$

Undergrad. Optics

(b.) A doubly refracting crystal has indices of refraction n_s and n_F for light polarized along the "slow" and "fast" axes, respectively. It is desired to cut a quarter wave plate from this material. The light is monochromatic and has angular frequency ω. What should be the thickness of the plate?

Number of wavelengths in distance x : $N = \dfrac{kx}{2\pi}$

Phase velocity : $v = \dfrac{c}{n} = \dfrac{\omega}{k}$ $\qquad N = \dfrac{n\omega x}{2\pi c}$

$v_F > v_S \implies n_F < n_S . \implies N_F < N_S$

Choose x so that $N_S - N_F = \dfrac{1}{4}$.

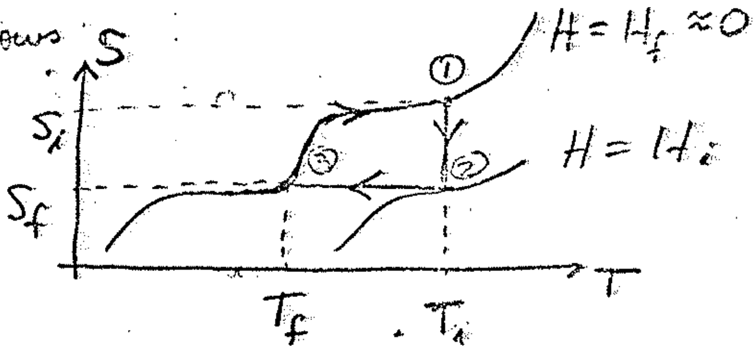
$$\frac{(n_s - n_F)\,\omega x}{2\pi c} = \frac{1}{4}$$

$$\text{or } x = \frac{\pi c}{2\omega(n_s - n_F)}$$

Thermal Phys. (Undergrad)

3.)

In an experiment to achieve ultralow temperatures by adiabatic demagnetization of a spin-½ paramagnetic salt the cycle may be represented on an entropy - temperature plot as follows:



The cycle ① → ② → ③ → ① is performed by first isothermally magnetizing the sample in a field H_i, by demagnetizing it adiabatically to a field $H_f \approx 0$, and by allowing small residual heat input (due to less than perfect thermal insulation) to slowly warm the sample back to T_i

(a) Assuming an electronic g-factor $g = 2$, complete the following table for the energy splitting $\Delta E = E(J_z = +\frac{1}{2}) - E(J_z = -\frac{1}{2})$ and population ratio $R = N(J_z = +\frac{1}{2}) / N(J_z = -\frac{1}{2})$ when the system is in states ①, ②, and ③

State	ΔE	$R = \frac{N(J_z = +\frac{1}{2})}{N(J_z = -\frac{1}{2})}$
①	≈ 0	1
②		
③		

(b) Assuming the population to be invariant in the process ② → ③, show that

$$\frac{H_i}{T_i} = \frac{H_f}{T_f}$$

(From ~~statistical~~ considerations only, calculate the change in entropy $\Delta S = S_i - S_f$ for an ensemble of N molecules. Assume only the lowest energy state to be populated at $T = T_f$ and assume all states to be energy degenerate at $H_f \simeq 0$.

(d) Based on the S-T graph given above, sketch the behavior of the heat capacity at constant field C_H as a function of T in the range of T as given above. Be sure to indicate the points T_i and T_f. Keeping in mind the residual heat leak, what value of the temperature would be most likely shortly after the demagnetization?

(e) In practice the field may not be reduced to zero, owing in part to the small but non-vanishing electronic dipole field. Estimate the ultimate temperature using the result of part (b) by estimating sensible values of H_i, H_f, and T_i.

Thermal Physics (Undergrad)

Solution

(a)

	ΔE	$N(+)/N(-)$
①	≈ 0	1
②	$2\mu_B H_i$	$e^{-2\mu_B H_i/kT_i}$
③	≈ 0	$e^{-2\mu_B H_f/kT_f}$

spin ½ } Boltzmann? distribution

(b) From above

$$e^{-2\mu_B H_i/kT_i} = e^{-2\mu_B H_f/kT_f}$$

$$\therefore \frac{H_i}{T_i} = \frac{H_f}{T_f}$$

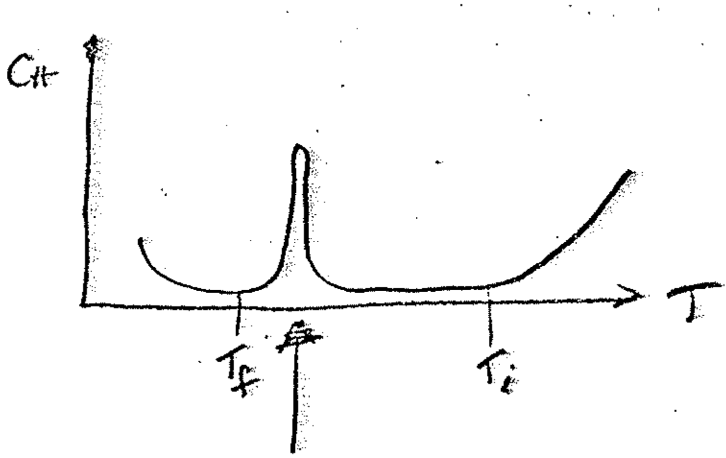
(c) $S = k \ln w$ w = number of states accessible to the system

$w_i = (2J+1)^N$

$w_f = 1$

$$\Delta S = Nk \ln(2J+1) - 0 = \boxed{Nk \ln 2} \text{ for } J = \tfrac{1}{2}$$

(d)



Most probable Temperature is at peak of heat capacity curve.

H_f is due to dipole fields, $H_f \approx \frac{\mu_0}{4\pi}\frac{m}{r^3}$; taking the magnetic moment to be one Bohr magneton, and taking r to be 5 Å,

≈ 100 gauss. $H_i \approx 10$-100 kilogauss (typical for large superconducting

(4.) The solution to prob 4. is easily seen without any computation, for, consistent with the allowed orbits of a mass in a gravitational field, stable orbits are ellipses.

So $\frac{\omega_{radial}}{\omega_{rotation}} = 1$

By computation we can use the method of small oscillations and write

Potential energy $= -\,G\frac{mM}{r}$

$G =$ grav. constant
$m =$ mass earth
$M =$ mass of Sun

Total energy $= \frac{1}{2}[m\dot{r}^2 + mr^2\dot{\theta}^2] - G\frac{mM}{r}$

Angular momentum $= mr^2\dot{\theta} \equiv J$

Thus; energy $= \frac{1}{2}\left[m\dot{r}^2 + \frac{J^2}{mr^2}\right] - G\frac{mM}{r}$

Expanding about original circular radius ρ_0 to second order

$$E = \frac{1}{2}\left\{m(\dot{\rho}-\dot{\rho}_o)^2 + \frac{J^2}{m}\left[\frac{-2}{\rho_o^3}(\rho-\rho_o) + \frac{(3)(2)}{(2)\rho_o^4}(\rho-\rho_o)^2\right]\right.$$

$$\left. + \frac{GmM}{\rho_o^2}(\rho-\rho_o) - \frac{2\,GmM}{(2)\,\rho_o^3}(\rho-\rho_o)^2 + \frac{J^2}{m\rho_o^2} - \frac{GmM}{\rho_o}\right\}$$

The stable orbit radius ρ_o is found by locating the minimum in the potential energy

$$\left(\frac{1}{2}\right)\frac{-2J^2}{m\rho_o^3} + \frac{GmM}{\rho_o^2} = 0 \qquad \rho_o = \frac{J^2}{m^2 MG}$$

The frequency of radial oscillations is found from second order terms (harmonic)

$$\left(\frac{1}{2}\right)\left(\frac{3J^2}{\rho_o^4 m}\right) - \frac{GmM}{\rho_o^3} \equiv \frac{1}{2} m\,\omega_R^2$$

But

$$J = m\rho_o^2\omega_o$$

So

$$\frac{1}{2}\left[\frac{3m^2\rho_o^4\omega_o^2}{\rho_o^4 m} - 2\frac{GmM}{\rho_o^3}\right] = \frac{1}{2} m\,\omega_R^2$$

and

$$\omega_R^2 = 3\omega_o^2 - 2\frac{GM}{\rho_o^3}$$

but

$$\rho_o^3 = \frac{MG}{\omega_o^2}$$

so

$$\omega_R^2 = 3\omega_o^2 - 2\omega_o^2 = \omega_o^2$$

5. Q.M. (Total)

Consider a single electron in its ground state orbit about an H^3 (tritium) nucleus, and suppose this nucleus beta decays, leaving behind an He^3 nucleus. Assume the beta decay takes place instantaneously; that is, the β-particle is relativistic and the time it takes to cross the electron's orbit is small compared to the characteristic time constant of the electron.

(a) What is the probability that the electron will be in a 1s state about the He^3 nucleus after the transition? Take the ground-state hydrogenic wave function to be $\Psi = \frac{1}{\sqrt{\pi}} \alpha^{3/2} e^{-\alpha r}$ where $\alpha = mZe^2/\hbar^2$.
CAPITAL

(b) In what other states may the electron find itself after the transition? (Ignore effects associated with spin, relativity, and nuclear recoil) Specify the allowed states by giving their quantum numbers.

c) What is the probability that the He^3 atom will be doubly ionized? ~~[illegible]~~ Specify your result in terms of ~~[illegible]~~ the amplitudes of the appropriate ~~basis~~ bound states, giving their

6. Spin waves (magnons) in a ferromagnetic solid are boson excitations described by the dispersion relation $\omega = Ak^2$, where ω is the angular frequency, k is the wave number, and A is a constant. Show that these excitations make a contribution to the heat capacity which is proportional to $T^{3/2}$, where T is the Kelvin temperature.

$$C_V = \frac{\partial \mathcal{U}}{\partial T} \quad ; \quad \mathcal{U} = \int_0^\infty u(\omega,T)\, d\omega$$

Energy per unit frequency interval at freq. ω and temp. T.

$$u(\omega,T) = \hbar\omega \cdot \frac{dN}{d\hat{N}} \cdot \frac{d\hat{N}}{d\omega}$$

Number of particles per state — Number of states per unit freq. interval

$$\frac{dN}{d\hat{N}} = \frac{1}{e^{\hbar\omega/kT} - 1} \qquad \text{Bose-Einstein distribution}$$

Volume of solid.

$$d\hat{N} = \frac{V d^3k}{(2\pi)^3} = \frac{V \cdot 4\pi k^2 dk}{(2\pi)^3} = \frac{V\, \omega^{1/2}\, d\omega}{4\pi^2 A^{3/2}}$$

$$\frac{d\hat{N}}{d\omega} = \frac{V\omega^{1/2}}{4\pi^2 A^{3/2}} \Longrightarrow u(\omega,T) = \frac{V\hbar}{4\pi^2 A^{3/2}} \cdot \frac{\omega^{3/2}}{\left(e^{\hbar\omega/kT} - 1\right)}$$

$$U = \frac{V\hbar}{4\pi^2 A^{3/2}} \int_0^\infty \frac{\omega^{3/2}\, d\omega}{e^{\hbar\omega/kT} - 1}$$

Let $x = \frac{\hbar\omega}{kT}$.

$$U = \frac{V\hbar}{4\pi^2 A^{3/2}} \left(\frac{kT}{\hbar}\right)^{5/2} \int_0^\infty \frac{x^{3/2}\, dx}{e^x - 1}$$

The integral is a constant.

Therefore $U \propto T^{5/2}$.

$$C_v = \frac{\partial U}{\partial T} \propto T^{3/2} \qquad \text{QED}$$