COMPREHENSIVE EXAMINATION #18
SPRING 1974

This Comprehensive Examination for Spring 1974 (#18) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half are graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook — you may get some credit for it!

If something is omitted from the statement of the problem or you feel there are ambiguities, get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books and papers, except pencil and bluebook, on the floor.

Some information you may find useful follows:

$\pi = 3.14159$ $\qquad$ $c \cong 3 \times 10^{8}$ m/sec

$h \cong 6.63 \times 10^{-34}$ joule-sec $= 4.0 \times 10^{-15}$ eV-sec

$k \cong 1.38 \times 10^{-23}$ joule/°K $= 8.6 \times 10^{-5}$ eV/°K

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\ln_e 1 = 0.000$ $\qquad$ $\displaystyle\int \frac{dx}{1 - x^2} = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| = \tanh^{-1} x \, (x^2 < 1)$

$\ln_e 10 = 2.303$ $\qquad$ $\displaystyle\int \frac{x\,dx}{(1 - x^2)^{3/2}} = \frac{1}{(1 - x^2)^{1/2}}$

$\ln_e 100 = 4.605$

$\ln_e 1000 = 6.908$

If you need a table of integrals or table of logarithms, ask the proctor.

1a) The "big bang" cosmological model should have left the universe filled with remnants of the primeval fireball — a sea of blackbody radiation in which all matter is immersed. Unfortunately, galactal noise obscures the signal at $\lambda > 10$ cm, so radio astronomy is used to detect the fireball remnants.

It is found that in stellar absorption spectra, cyanogen molecules (CN) in intergalactal space which are also in the radiation sea, absorb at $\lambda = 2.6$ mm, and are thus raised to their first excited state. Those in the first excited state absorb at $\lambda = 3.8$ mm. The absorption ratio is measured to be

$$\frac{\text{absorption (3.8 mm)}}{\text{absorption (2.6 mm)}} = 0.135$$

Calculate the temperature of the fireball remnant.

1b) A new possible fourth law of thermodynamics says that there exists an ultimate temperature which cannot be exceeded, regardless of the energy absorbed by a system. Consider one mole of molecular hydrogen confined in a fixed volume to which energy is added continuously without loss.

In the following table provide the missing information about the temperature T and the specific heat C_v for the system. Make a calculation for each step; if you cannot, at least indicate whether C_v is falling, rising, or is constant.

		C_v	$T(^\circ K)$
a)	gas warm, specific heat nearly constant	______	~500
b)	dissociation begins	______	______
c)	dissociation almost complete; we now have two moles of atomic hydrogen	______	______
d)	ionization begins	______	______
e)	ionization almost complete; we now have two moles of protons and two moles of electrons	______	______
f)	collisions produce electron-positron pairs	______	______
g)	proton-proton collisions begin to produce pions	______	______
h)	pions abundant; p-p collisions begin to produce nucleon-anti-nucleon pairs. This is the ultimate temperature.	______	______

2. X-rays of frequency ω are impinging on a metal plate. The X-rays make an angle θ with respect to the normal of the plate. Assuming that there are N electrons per unit volume in the metal, and that these are weakly bound to the ions and thus execute simple harmonic motion with the frequency ω of the x-rays,

a) calculate the x-ray induced polarizability of the metal.

b) what is the dependence of the critical angle for reflection on the frequency ω of the x-rays?

c) for what frequency range does one obtain total reflection at all incident angles?

3. A rocket is shot out from the earth into interstellar space. Except for a short time in the beginning, the acceleration of the rocket as measured by the passengers is a constant, a'. The rocket has been aimed at a star at a fixed distance D from the earth, and moves on a straight line. According to a clock inside the rocket, how long will it take to get to the star?

4. A very long iron plunger of permeability μ is inserted part way into a long closely-fitting solenoid of n turns per unit length, so that one end is near the solenoid's center, and the other is in an essentially field-free region. The cross-section of the solenoid (and plunger) is A, and it carries a steady current i. In terms of the parameters given, determine the force pulling the plunger further into the solenoid.

5a) A conducting sphere of radius a carries a charge Q. The sphere is cut in half, with the hemispheres remaining in contact. Calculate the force of repulsion between the hemispheres.

b) Three similar isolated neutral conductors are arranged so that each is perfectly symmetrical with respect to the other two. A wire from a battery of unknown voltage is touched to each in turn. The charges on the first two conductors are found to be Q_1 and Q_2. What is the charge on the third?

6. An electrostatic lens produces an electric field through which a beam of charged particles will pass. With the z-axis defined to coincide with the longitudinal axis of the lens, the potential between the electrodes is given by

$$V(x,y,z) = K(x^2-y^2) \quad , \text{ where } K = \text{constant.}$$

a) Sketch the equipotentials in the x-y plane.

b) Solve the equations of motion for an electron having an initial velocity $\vec{v} = v_o\hat{z}$ and an initial position $\vec{r}_o = x_o\hat{x} + y_o\hat{y}$ in the region between the electrodes.

c) If K = 1000 volts/m^2 and the electrons have 10 eV of kinetic energy, calculate the focal length of the lens. (The focal length is the distance along the z-axis from the initial position of the electron to the point at which it first crosses the y-axis.)

1a The "big bang" cosmological model should have left the universe filled with remnants of the primeval fireball — a sea of blackbody radiation in which all matter is immersed. Unfortunately, galactic noise obscures the signal at $\lambda > 10$ cm, so radio astronomy is used to detect the fireball remnation.

It is found that in stellar absorption spectra, cyanogen molecules (CN) in space, in their ground state, immersed in this radiation, absorb at $\lambda = 2.6$ mm, and are thus raised to their 1st excited state. Those in this excited state absorb at 3.8 mm. The absorption ratio is found to be

$$\frac{\text{abs}(3.8\text{ mm})}{\text{abs}(2.6\text{ mm})} = 0.135$$

What is the blackbody temperature of the fire ball remnant?

---

1. The absorption is proportional to the number of CN molecules in any state,

so $$\frac{n_1}{n_0} = 0.135$$

2. The energy of the 1st excited state is $\mathcal{E} = \frac{hc}{\lambda} = \frac{4\times10^{-15}\text{ eV-sec}\times3\times10^{8}\text{ m/sec}}{2.6\times10^{-3}\text{ m}} = 4.6\times10^{-4}\text{ eV}$

3. Since $\frac{n_1}{n_0} = e^{-\frac{\mathcal{E}}{kT}}$, $$T = \frac{\mathcal{E}}{-k\ln\left(\frac{n_1}{n_0}\right)} = \frac{4.6\times10^{-4}\text{ eV}}{-8.6\times10^{-5}\frac{\text{eV}}{{}^\circ\text{K}}\ln(0.135)} = \frac{4.6\times10^{-4}}{-8.6(-2)\times10^{-5}} = \underline{\underline{2.7^\circ\text{K}}}$$

need $\begin{cases}\ln \text{ tables} \\ c \\ h\end{cases}$

D) A new possible fourth law of thermodynamics says that there exists an ultimate temperature which cannot be exceeded, regardless of the energy absorbed by a system.

Consider one mole of molecular hydrogen confined in a fixed volume, to which energy is added continuously.

In the following table, provide all the information you can about the temperature T and the specific heat C_v for the system. Make a calculation for each step if you can; where you cannot, at least indicate whether C_v is constant or rising.

	C_v	T(°K)
a) gas warm, specific heat nearly constant.	$\frac{5}{2}R$	~500
b) dissociation begins.	rising	5000
c) dissociation virtually complete; we now have two moles of atomic hydrogen.	3 R	10^4
d) ionization of atoms begins.	rising	10^4
e) ionization virtually complete; we now have two moles of protons and two moles of electrons.	6 R	10^5
f) particle collisions produce electron-positron pairs	rising	10^9
g) proton-proton collisions begin to produce pions.	rising	10^{11}
h) pions abundant; p-p collisions begin to produce proton-anti proton pairs. This is the ultimate temperature.	rising	10^{12}

Not given.

need k

Since the expt. proceeds at constant volume, there is no PdV work; the added energy goes to increase the internal energy ΔU. The definition $C_v = \left(\frac{\partial U}{\partial T}\right)_v$ implies that

1. When $C_v \approx$ constant, $\Delta U = C_v \Delta T$ appears as an increase in kinetic energy (rise in temp).

2. When C_v increases, energy is being used for dissociation, ionization, particle production, not to raise the temperature (kinetic energy).

a) When the molecules are whole, each degree of freedom has a specific heat of $\frac{1}{2}R$ per mol, so $C_v = \frac{5}{2}R$.

b) The gas is starting to change into a mixture of molecular and atomic hydrogen; an atom has 3 degrees of freedom, so $\underline{C_v \text{ is rising}}$. While dissociation energy is not available for increasing the temperature, the dissociation will be virtually complete. Since there would be a significant number of particles with enough energy to initiate dissociation by collision at $kT \sim \frac{1}{10} E_{diss}$, we can use this as a rough estimate of an ignition temperature.

now $E_{diss} \approx 4\ eV$ and $0.025\ eV = 300^\circ K$

so $kT = 0.4 \times \frac{300}{.025} = 4800^\circ K \approx \underline{5000^\circ K}$

c) at $KT = E_{dissoc} \sim \underline{10^4\ ^\circ K}$ we have two moles of atoms, each with 3 degrees of freedom, so $\underline{C_v = 3R}$

d) as the atoms dissociate, we make more free particles, C_v rises.

$$kT = 1.36\ eV \times \frac{300}{0.025} \approx 16{,}000^\circ$$ is the ignition temperature estimate.

e) at $KT \sim \underline{10^5\ ^\circ K}$, we have 2 moles of protons, 2 of electrons, $\underline{C_v \approx 6R}$ and remains constant up to $\underline{\sim 10^8\ ^\circ K}$

f) as we begin to produce pairs, which requires $E \sim 2mc^2 \sim 10^6\ eV$, we make more particles, so $\underline{C_v \text{ is rising}}$, and the ignition temp is

$$kT \sim 10^5 \times 12{,}000^\circ \approx \underline{10^9\ ^\circ K}$$

g) as we begin to produce pions, requiring $E = m_\pi c^2 \approx 135$ MeV, we have more particles, So C_V is rising, and the equation temp is

$$KT \sim 1.35 \times 10^7 \times 12{,}000 \cong \underline{1.6 \times 10^{11}\ {}^\circ K}$$

h) The ultimate temperature is $\sim \underline{10^{12}}\ {}^\circ K$; C_V is rising, because we start making a few exotic particles. However, the production of many pions acts as a thermostat, a governor for the temperature. Adding more energy only increases numbers of particles, not their KE.

Comp Exam E+M.

2

X rays with frequency ω are impinging on a metal plate. The X rays make an angle θ with respect to the normal of the plate.
Assume that there are N electrons per unit Volume in the metal. The electrons are weakly bound to the ions and execute simple harmonic motion with the frequency ω of the X-rays.

a) Calculate the X-ray induced polarizability of the metal.

b) What is the dependence of the critical angle for reflection on the frequency ω of the X-rays?

c) For what frequency range does one obtain total reflection at all incident angles?

Solution E+M.

for X-rays $\vec{E} = \vec{E}_0 e^{-i\omega t}$.

thus
$$m\ddot{x} = eE = eE_0 e^{-i\omega t}.$$

electrons execute simple harmonic motion

$$\ddot{x} + \omega^2 x = 0 \qquad x = x_0 e^{-i\omega t}.$$

thus
$$\omega^2 x_0 = -\frac{eE_0}{m}$$

max induced dipole moment is
$$e\,x_0 = -\frac{e^2 E_0}{m\omega^2}.$$

for a single ion-electron pair.

Polarization of metal is $\vec{P} = -N\frac{e^2}{m\omega^2}\vec{E}_0$

and thus the polarizability

a) $$\boxed{\alpha = -\frac{Ne^2}{m\omega^2}}$$

Solution E+M

$$\vec{D} = \varepsilon\vec{E} = \vec{E} + 4\pi\vec{P} = \vec{E}(1 + 4\pi\alpha)$$

index of refraction.

$$n^2 \equiv \varepsilon = 1 - \frac{4\pi N e^2}{m\omega^2}.$$

From Snell's law.

$$\sin\theta_i = n \sin\theta_r$$

b) $$\sin\theta_{cr} = n = \left(\frac{4\pi N^2 e^2}{m\omega^2}\right)^{1/2} \equiv \frac{\omega_p}{\omega}$$

$$\omega_p = \left(\frac{4\pi N^2 e^2}{m}\right)^{1/2} \text{ - (plasma frequency)}$$

Thus for

c) $\omega < \omega_p$ total ~~(internal)~~ reflection at all angles.

(3) Comp Exam Relativity

A rocket is shot out from the earth into interstellar space. Except for a short time in the beginning, the acceleration of the rocket as measured by the passengers is constant, a'. The rocket has been aimed at a star a fixed distance D from the earth, and moves on a straight line. According to a clock inside the rocket, how long will it take to get to the star? ~~Denote the constant distance and acceleration by D and a' respectively.~~

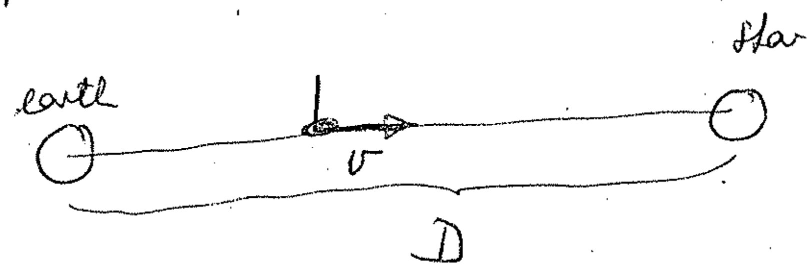
Solution

frame fixed with star $\quad x, t \quad u_x \quad a_x$

frame fixed in rocket $\quad x', t' \quad u_x' \quad a_x'$

$$x = \gamma(x' + vt')$$
$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

v = speed of rocket with respect to earth



$$u_x = \frac{dx}{dt} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}$$

to get relationship between accelerations transform.

$$\frac{u_x + du_x}{dt} - \frac{u_x}{dt} = a_x$$

$$a_x = \frac{a_x'}{\gamma^3\left(1 + \frac{u_x' v}{c^2}\right)^3} \qquad \text{for } \frac{dx'}{dt'} = 0$$

$$a_x = \frac{a_x'}{\gamma^3} \qquad \text{also} \quad dt = \gamma\, dt'$$

Total time elapsed $T' = \int dt' = \int \frac{dt}{\gamma} = \int \frac{1}{\gamma a_x} dv$

$$T' = \frac{1}{a_x'} \int_0^{v_f} \gamma^2 dv = \frac{1}{a_x'} \int_0^{v_f} \frac{dv}{1 - \frac{v^2}{c^2}}$$

$$T' = \frac{1}{a'} \frac{1}{2} \log \left| \frac{1 + \frac{v_f}{c}}{1 - \frac{v_f}{c}} \right|$$

find v_f in terms of $\underline{\ \ D}$

$$D = \int v\,dt = \int \frac{v\,dv}{a} = \frac{1}{a'}\int_0^{v_f} \frac{v\,dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$D = \frac{1}{a'}\left[\frac{1}{\sqrt{1-\frac{v_f^2}{c^2}}} - 1\right]$$

solve for v_f and substitute into expression for T'

Integrals needed.

$$\int \frac{dy}{1-y^2} = \frac{1}{2}\log\left|\frac{1+y}{1-y}\right| = \tanh^{-1} y \quad (y^2<1)$$

$$\int \frac{y\,dy}{K^3} = \frac{1}{K} \qquad K = (b^2-y^2)^{1/2}$$

4

Two long iron plungers of permeability μ are inserted in a very long, closely fitting solenoid of n turns per unit length. The cross-section of the solenoid is A, and it carries a current i. If the plungers are separated a short distance, determine (in terms of the parameters given) the force tending to draw them together.

Info required to solve problem:

1. $U = \frac{1}{2}\int \vec{B}\cdot\vec{H}\,dv$ $\qquad \vec{B} = \mu_0\vec{H}$ in a vacuum.
2. field inside a long solenoid is uniform. $\qquad = \mu\mu_0\vec{H}$ in iron
3. $F = -\frac{\partial U}{\partial x}$ $\qquad H = ni$ for a solenoid.

w or w/o iron

For a volume $A\Delta x$, $\quad U = \frac{1}{2} ABH\Delta x \qquad$ so $\Delta U = U_{iron} - U_{gap} =$ [redacted]

and $F_x = -\frac{\partial U}{\partial x}$ [redacted]

$$\Delta U = \frac{1}{2} A\Delta x \, B_{iron} H - \frac{1}{2} A\Delta x \, B_{gap} H$$

$$= \frac{1}{2} A\Delta x \left(\mu\mu_0 H^2 - \mu_0 H^2\right) = \frac{1}{2} A (ni)^2 \mu_0 (\mu - 1)\,\Delta x$$

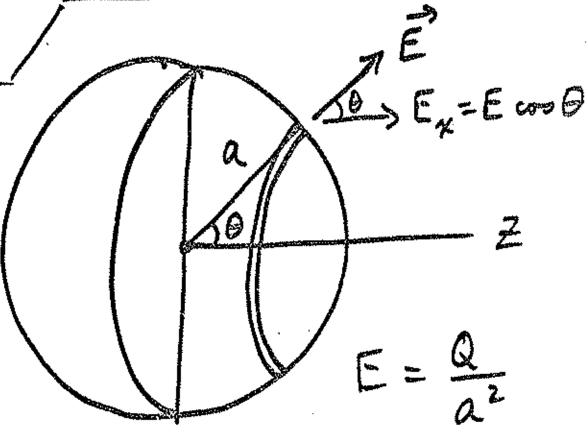
$$\text{so } F_x = -\frac{\partial U}{\partial x} = \underline{\frac{\mu_0}{2} A n^2 i^2 (\mu - 1)} \qquad \text{MKS}$$

$$= \underline{\underline{\frac{2\pi}{c^2} A n^2 i^2 (\mu - 1)}} \qquad \text{Gaussian.}$$

Graduate E + M

5

(a) A conducting sphere of radius $\underline{a}$ carries charge Q. The sphere is cut in half, with the hemispheres remaining in contact. Calculate the force of repulsion between the hemispheres.



$$F_x = \int \tfrac{1}{2}\sigma E_x \, dA \qquad \sigma = \frac{Q}{4\pi a^2}$$

$$= \frac{1}{2}\cdot\frac{Q}{4\pi a^2}\int_{\theta=0}^{\pi/2} E\cos\theta \cdot a^2 \cdot 2\pi\sin\theta \, d\theta \qquad E = \frac{Q}{a^2}$$

$$= \frac{1}{4}Q\cdot\frac{Q}{a^2}\cdot\frac{\cos^2\theta}{2}\Big|_{\pi/2}^{0} = \frac{Q^2}{8a^2}$$

5

(d) Three similar isolated neutral conductors are arranged so that each is perfectly symmetrical with respect to the other two. A wire from a battery of unknown voltage is touched to each in turn. The charges on the first two conductors are found to be Q_1 and Q_2. What is the charge on the third?

$V_i = \sum_j p_{ij} q_j$, where the p_{ij} are the coefficients of potential.

Let V = battery voltage. By symmetry, $\begin{cases} p_{11} = p_{22} = p_{33} \equiv p \\ p_{21} = p_{31} = p_{32} \equiv p' \end{cases}$

Then
$$V = p_{11}Q_1 \longrightarrow V = pQ_1 \longrightarrow p = V/Q_1$$
$$V = p_{21}Q_1 + p_{22}Q_2 \longrightarrow V = p'Q_1 + pQ_2 \rightarrow p' = \frac{V}{Q_1}\left(1 - \frac{Q_2}{Q_1}\right)$$
$$V = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3 \longrightarrow V = p'(Q_1 + Q_2) + pQ_3$$

$$V = \frac{V}{Q_1}\left(1 - \frac{Q_2}{Q_1}\right)(Q_1 + Q_2) + \frac{V}{Q_1}Q_3 \Longrightarrow Q_3 = Q_1 - \left(1 - \frac{Q_2}{Q_1}\right)(Q_1 + Q_2) = \frac{Q_2^2}{Q_1}$$

6 Undergraduate E+M

An electrostatic quadrupole lens is made from four long electrodes of hyperbolic cross section, producing an electric field through which a beam of charged particles will pass. With the z-axis defined to coincide with the ~~symmetry~~ longitudinal axis of the lens, the potential between the electrodes is given by

$$V(x,y,z) = K\cdot(x^2-y^2), \quad \text{where } K \text{ is a constant.}$$

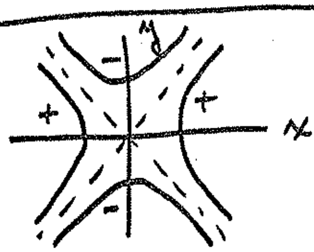
a) Sketch the equipotentials in the x-y plane.

b) Solve the equations of motion for an electron having an initial velocity $\vec{v}_0 = v_0\hat{z}$ and an initial position $\vec{r}_0 = x_0\hat{x} + y_0\hat{y}$ in the region between the electrodes.

c) If $K = 1000$ volts/meter^2 and the electrons have 10 eV of kinetic energy, calculate the focal length of the lens. The focal length is the distance along the z-axis from the initial position of the electron to the point at which it first crosses the y-axis.

---

a)



b) $m\ddot{x} = -eE_x = e\dfrac{\partial V}{\partial x} = 2eKx \Rightarrow x = x_0\cosh\omega t$

$m\ddot{y} = -eE_y = e\dfrac{\partial V}{\partial y} = -2eKy \Rightarrow y = y_0\cos\omega t$

$m\ddot{z} = 0 \Rightarrow z = v_0 t.$ where $\omega \equiv \sqrt{\dfrac{2eK}{m}}$.

c) $f = v_0 t = v_0\dfrac{\pi}{\omega} = \sqrt{\dfrac{2eV_0}{m}}\cdot\pi\sqrt{\dfrac{m}{2eK}} = \pi\sqrt{\dfrac{V_0}{K}}$

$V_0 = 10$ volts

$K = 1000$ volts/meter^2

$f = \pi\sqrt{\dfrac{10}{1000}}$ meters $= 31.4$ cm