#17 - #19

Winter 1974

This Comprehensive Examination for Winter 1974 (#17) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half are graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook -- you may get some credit for it!

If something is omitted from the statement of the problem or you feel there are ambiguities, get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books and papers, except pencil and bluebook, on the floor.

Some information you may find useful follows:

$\pi = 3.14159$

$h \cong 6.63 \times 10^{-34}$ joule-sec.

Poisson distribution $P(n) = \frac{1}{n!} (\bar{n})^n e^{-\bar{n}}$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\ln_e 1 = 0.000$

$\ln_e 10 = 2.303$

$\ln_e 100 = 4.605$

$\ln_e 1000 = 6.908$

If you need a table of integrals, ask the proctor.

Integrating,

$$\ln \phi = -\int \frac{(2p + ia)}{(p^2+\hbar^2k^2)}\,dp = -2\int \frac{p\,dp}{p^2+\hbar^2k^2} - ia\int \frac{dp}{p^2+\hbar^2k^2}$$

now $\int \frac{x\,dx}{ax^2+b} = \frac{1}{2a}\ln(ax^2+b)$

and $\int \frac{dx}{ax^2+b} = \frac{1}{\sqrt{ab}}\tan^{-1} x\sqrt{\frac{a}{b}}$

$$= -\ln(p^2+\hbar^2k^2) - \frac{ia}{\hbar k}\tan^{-1}\frac{p}{\hbar k} + \ln A$$

$$\ln \phi(p^2+\hbar^2k^2) = -\frac{ia}{\hbar k}\tan^{-1}\frac{p}{\hbar k} + \ln A$$

$$\phi(p) = \frac{A}{(p^2+\hbar^2k^2)}\, e^{-\frac{ia}{\hbar k}\tan^{-1}\frac{p}{\hbar k}}$$ are the eigenfunctions

In order that the $\phi(p)$ be well-behaved (single-valued) $\phi(p)$ must be unchanged as $\tan^{-1}\frac{p}{\hbar k} \to \tan^{-1}\frac{p}{\hbar k} + \pi$ since $\tan x$ has period π.

Then, we require $e^{-i\frac{a}{\hbar k}\tan^{-1}\frac{p}{\hbar k}} = e^{-i\frac{a}{\hbar k}\left(\tan^{-1}\frac{p}{\hbar k} + \pi\right)}$ so that we need $e^{-i\frac{a\pi}{\hbar k}} = 1$

this will be true if $\frac{a\pi}{\hbar k} = 2\pi n \qquad n = 1, 2, 3 \ldots$

so $\hbar k = \frac{a}{2n}$ and $\frac{\hbar^2k^2}{2m} = \frac{a^2}{8mn^2} = \frac{me^4}{2\hbar^2 n^2}$ since $a = \frac{2me^2}{\hbar}$

so that $E = -\frac{\hbar^2k^2}{2m} = -\frac{me^4}{2\hbar^2n^2}$, the usual eigenvalues for the $l=0$ states of the hydrogen atom.

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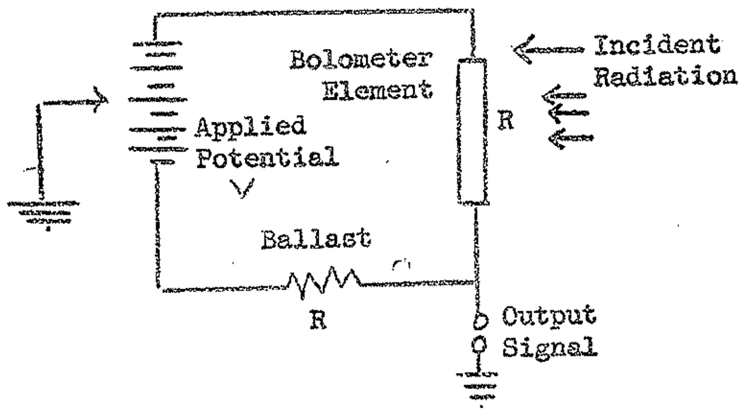

When radiation strikes a bolometer it heats up, changing its resistance. This change can be used to detect and measure radiation in the following way: a bolometer (a 1 mm^2 strip of platinum whose resistance is $R_o = 50\Omega$) and its surroundings are initially at temperature T_o. By applying the potential V (but no radiation) a bias current is caused to flow; this current reaches the steady-state value I when it has heated the bolometer to a steady-state resistance R at temperature T. (Platinum has a linear resistance coefficient $\alpha = 3.8 \times 10^{-3} (^oC)^{-1}$; the ballast resistance is usually an identical element to balance out any drift. The ballast is of course shielded from radiation.) The main loss is by conduction to the supports; the conduction cooling constant is typically $K = 1.2 \times 10^{-5}$ watts/^oC. After the steady state is reached, power P of continuous radiation is allowed to strike the element, heating it slightly so that R becomes R + r and I becomes I-i ($r \ll R$, $i \ll I$) at the new equilibrium. If $T-T_o$ is 25^oC, and the element reflects little or no radiation,

a) Calculate the sensitivity of the system, i.e., show that

$$S \equiv \frac{iR}{P} = \frac{\alpha R_o}{2}\left(\frac{T-T_o}{RK}\right)^{\frac{1}{2}}$$

b) If the lowest acceptable (signal/Johnson noise) ratio is one, calculate the smallest radiant power level which can be detected when a low-pass filter passing 0-10 Hz is placed across the output.

2. Positronium is a bound state of an electron and a positron. The Hamiltonian for the system in a magnetic field $\mathcal{H}$ can be written in the form

$$H = H_o + A(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + \mu_B \mathcal{H} (\sigma_{1z} - \sigma_{2z})$$

where H_o contains kinetic energies and central force potentials, A is a constant and $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are vectors whose components are the Pauli matrices. The index 1 refers to the electron and the index 2 to the positron.

a) Find the value of A if in zero magnetic field the 1^1S and the 1^3S states are separated by 2×10^5 M Hz with the singlet state the lowest.

b) Discuss the principles which restrict the 1^1S and 1^3S ground states to decay primarily by two-and three-quantum emission, respectively.

c) Find the eigenvalues of the 1S states for non-vanishing magnetic field

3. Nearly monochrometic light from some source is viewed through a polarizing analyzer A. There are orientations of A for which maximum intensity is transmitted but none which give zero intensity. A quarter-wave plate is appropriately installed and it is observed that there is still no orientation of A which gives zero intensity, but a rotation of A by 30° from the position which formerly gave maximum intensity restores this condition.

   a) Describe in some detail how a polarizing analyzer and a quarter-wave plate work.

   b) What is the position and orientation of the quarter-wave plate with respect to analyzer A?

   c) From the information given what is your best conclusion about polarization of the incident light from among the following possibilities:

unpolarized	completely elliptical
partly plane	partly circular
completely plane	completely circular
partly elliptical	

   Give your reasons.

4. In the Lorentz model of an atom, an electron (mass M, charge $-e$) is considered to be bound to a heavy nucleus by a restoring force $\vec{F} = M\omega_o^2\vec{r}$, where $\vec{r}$ denotes the position of the electron with respect to the nucleus and ω_o is the natural frequency of linear oscillations of the electron. The atom is driven continuously by a linearly polarized monochromatic electromagnetic plane wave having frequency ω. Assume that a small radiation-reaction force

$$\vec{F}' = \frac{2e^2}{3c^3}\frac{d^3\vec{r}}{dt^3}$$

acts upon the electron. Calculate the total cross section $\sigma(\omega)$ for the scattering of radiation by this simple atom. After finding a general expression for σ, simplify your results to describe two important special cases:

   a) $\omega \gg \omega_o$ (Thomson cross section σ_T for a free electron; independent of frequency)

   b) $\omega = \omega_o$ (resonance cross section σ_R).

Estimate the numerical values in (a) and (b), assuming viable light in case (b).

HINT: In gaussian units, the Larmor power formula

$$P = \frac{2e^2}{3c^3}\left|\frac{d^2\vec{r}}{dt^2}\right|^2$$

gives the total instantaneous power radiated by an accelerating electron.

5a) A cubic box of volume $V = \ell^3$ contains 10^{20} atoms. Estimate the value of [illegible] for which on the average, only one atom in the box will go from wall to wall without suffering a collision with another atom.

b) A vessel contains an ideal gas at standard temperature and pressure. Calculate the probability that a part of the vessel of volume [illegible] $\times 10^{-2}$[illegible] cubic meters, contains no molecules.

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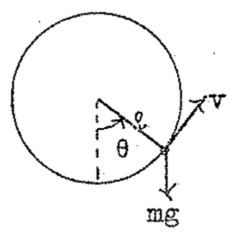
6a) A particle of mass m and weight W = mg is constrained to move in a vertical circle of radius ℓ under the influence of the gravitational force after being set in motion and released. If the particle obeys the laws of quantum mechanics, is the angular momentum conserved? Prove your statement.



b) Using the Heisenberg Uncertainty relation $pr \sim \hbar$ for the "best" localization of an electron in orbit, and a variational principle, estimate the binding energy (in eV) of a K-shell electron in carbon. Neglect relativistic effects.

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Before the incident radiation hits, the cooling rate must equal the joule heating in the element

$K(T-T_0) = I^2 R$ but since $R = R_0[1+\alpha(T-T_0)]$, $T - T_0 = \frac{R-R_0}{\alpha R_0}$

① So $K\left(\frac{R-R_0}{\alpha R_0}\right) = I^2 R$ and $2IR = V$ ②

When F watts fall steadily on the element, $R \to R+r$ and $I \to I-i$

③ So $K\left(\frac{R+r-R_0}{\alpha R_0}\right) = (I-i)^2(R+r) + F$ and $(I-i)(2R+r) = V$ ④

In order to calculate the sensitivity, we need $S = \frac{iR}{F}$.

We simplify by subtracting ① from ③

$$\left[\frac{K}{\alpha R_0} - (I-i)^2\right] r = \left[(I-i)^2 - I^2\right] R + F$$

and now we can eliminate r by equating ② and ④; $r = \frac{2i}{(I-i)} R$
and substituting;

$$\frac{K \cdot 2iR}{\alpha R_0 (I-i)} \cancel{-2IiR} + 2i^2 R = \cancel{-2iIR} + i^2 R + F$$

12 $$\frac{2(iR)K}{\alpha R_0 (I-\cancel{i})} + \cancel{i^2 R} = F$$ since $i \ll I$, i^2R can be neglected

$$\text{So } S = \frac{iR}{F} = \frac{\alpha R_0 I}{2K} = \frac{\alpha R_0}{2}\left(\frac{T-T_0}{RK}\right)^{1/2} = (3.8\times10^{-3}\ {}^\circ C^{-1}) \times 50\,\Omega \left(\frac{25\,{}^\circ C}{55\,\Omega \times 1.2\times10^{-5}\ \frac{W}{{}^\circ C}}\right)$$

$$= 18.5\ \frac{\text{volts}}{\text{watt}}$$

For $\Delta f = 10$ cps, $V_{RMS} = 2\sqrt{kTR(\Delta f)} = 2\sqrt{1.38\times10^{-23} \times 300 \times 55 \times 10} = 2.88\times10^{-9}$

8 So $P_{min} = \frac{2.88\times10^{-9}\ \text{volts}}{18.5\ \frac{\text{volts}}{\text{watt}}} = \underline{1.55\times10^{-10}\ \text{watts}}$

Grad QM Problem

Solution (2)

$1\,{}^3S$ $\quad \psi_{1,1} = \alpha_1 \alpha_2$

$\psi_{1,0} = \frac{1}{\sqrt{2}}(\alpha_1 \beta_2 + \beta_1 \alpha_2)$

$\psi_{1,-1} = \beta_1 \beta_2$

$1\,{}^1S$ $\quad \psi_{00} = \frac{1}{\sqrt{2}}(\alpha_1 \beta_2 - \beta_1 \alpha_2)$

with $\quad \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

eigenvalues of $\quad A(\vec{\sigma}_1 \cdot \vec{\sigma}_2)$

write $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}$

one gets

$$E_{0,0} = -3A \qquad E_{1,0} = E_{1,\pm 1} = A.$$

thus ${}^1S - {}^3S$ splitting is

$$\Delta E = 4A \qquad \text{with } A \text{ positive}$$

a) $\quad \frac{4A}{\hbar} = 2 \cdot 10^5\, MHz \qquad A = 5 \times 10^{-17}$ erg.

1) Single-quantum decay is impossible because linear momentum cannot be conserved.
Two quantum decay means that the two photons must go off in opposite directions. The total angular momentum of the two photons must be 0 or 2. Hence singlet state can decay by two photon emission. For decay of triplet state three photons are needed to conserve linear and angular momentum.

c). For $\mathcal{H} \neq 0$ one must solve.

$$
\begin{array}{c}
1,1 \rightarrow \\
1,0 \rightarrow \\
1,-1 \rightarrow \\
0,0 \rightarrow \\

\end{array}
\begin{array}{c}
\begin{pmatrix}
A-E & 0 & 0 & 0 \\
0 & A-E & 0 & 2\mu_B\mathcal{H} \\
0 & 0 & A-E & 0 \\
0 & 2\mu_B\mathcal{H} & 0 & -3A-E
\end{pmatrix} \\
\begin{array}{cccc}
1,1 & \quad 1,0 & \quad 1,-1 & \quad 0,0
\end{array}
\end{array}
= 0
$$

The solutions are

$$E_{1,\pm 1} = A \qquad E_{\left\{ 0,0 \atop 1,0 \right\}} = -A \pm 2\left[A^2 + \mu_B^2 \mathcal{H}^2\right]^{1/2}$$

$$(\psi_1 = \psi_{1,1} \quad \psi_2 = \psi_{1,-1})$$

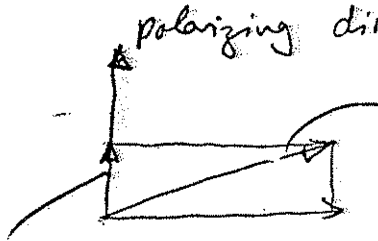
$$(\psi_3 = y\,\psi_{1,0} + \left[-2 + 2\sqrt{1+y^2}\right]\psi_{00}, \quad \psi_4 = \left[2 - 2\sqrt{1+y^2}\right]\psi_{1,0} + y\,\psi_{0,0}$$

$$y = 2\mu_B \mathcal{H}/A$$

Undergrad Optics Problem

Solution.

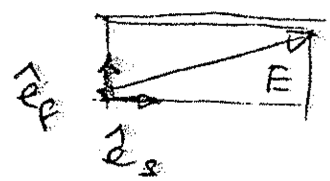
a)



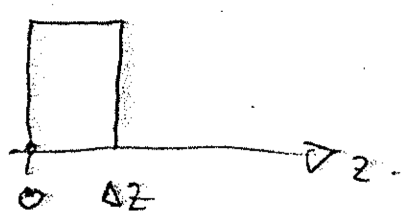
This component passes through analyzer.

quarter wave plate.

has a fast axis (smaller index of refraction) and a slow axis (larger index of refraction) The two axes are perpendicular to each other



$$E(0,t) = e^{i\omega t}\left[\hat{e}_s A_s e^{i\phi_s} + \hat{e}_f A_f e^{i\phi_f}\right]$$



$$E(z,t) = e^{i\omega t}\left[\hat{e}_s A_s e^{i(\phi_s - n_s \omega z/c)} + \hat{e}_f A_f e^{i(\phi_f - n_f \omega z/c)}\right]$$

phase retardation of E_s relative to E_f

$$\Delta\phi = (n_s - n_f)\frac{\omega \Delta z}{c} = (n_s - n_f)\, 2\pi \frac{\Delta z}{\lambda_{\dots}}$$

for $\Delta\varphi = \frac{\pi}{2}$.

linearly polarized light is converted into elliptically polarized light or vice-versa.

b). quarter wave plate in front of polarizer with fast axis aligned with polarization direction of A.

c). Since max is seen but no zero incoming light is

1) partly plane polarized
2) elliptically polarized
3) partly elliptically polarized.

1) is excluded because quarter wave plate would not produce a change in position of max.
2) is excluded because rotation of A would produce zero intensity
3) is correct. quarter wave plate would produce partly plane polarized light with max along fast axis which is rotated from semimajor axis of ellipse by 30°

4) $m\ddot{x} + m\omega_0^2 x - \frac{2}{3}\frac{e^2}{c^3}\dddot{x} = -eE_x$, where $\vec{E}$ is along x-axis.

$$E_x = E_0 e^{-i\omega t} \; ; \; x = x_0 e^{-i\omega t}$$

$$m(-\omega^2)x_0 e^{-i\omega t} + m\omega_0^2 x_0 e^{-i\omega t} - \frac{2e^2}{3c^3}(i\omega^3)x_0 e^{-i\omega t} = -eE_0 e^{-i\omega t}$$

$$x_0 = \frac{eE_0/m}{(\omega^2 - \omega_0^2) + i\cdot\frac{2e^2}{3mc^3}\omega^3}$$

$$\langle P_{radiated}\rangle = \sigma\cdot\left\langle\frac{P_{incident}}{A}\right\rangle$$

$$\left\langle\frac{P_{incident}}{A}\right\rangle = \frac{|E_0|^2}{8\pi}c \quad ; \quad \langle P_{radiated}\rangle = \frac{2e^2}{3c^3}\cdot\frac{1}{2}\omega^4|x_0|^2$$

$$\sigma = \frac{\frac{e^2}{3c^3}\omega^4|x_0|^2}{\frac{c}{8\pi}|E_0|^2} = \frac{8\pi e^2}{3c^4}\omega^4\cdot\left|\frac{e/m}{(\omega^2-\omega_0^2) + i\cdot\frac{2e^2}{3mc^3}\omega^3}\right|^2$$

$$\sigma = \frac{\left(\frac{8\pi e^4\omega^4}{3m^2c^4}\right)}{(\omega^2-\omega_0^2)^2 + \frac{4e^4\omega^6}{9m^2c^6}}$$

(a) $\omega \gg \omega_0$; $\frac{d\sigma}{d\omega} = 0$. $\quad \sigma_T = \frac{8\pi e^4}{3m^2c^4} \approx 10^{-24}\,\mathrm{cm}^2$

(b) $\omega = \omega_0$. $\quad \sigma_R = \frac{8\pi e^4\omega_0^4}{3m^2c^4}\cdot\frac{9m^2c^6}{4e^4\omega_0^6} = 6\pi\frac{c^2}{\omega_0^2} \approx 10^{-9}\,\mathrm{c}$

QM (V)

a) A particle of mass m and weight $W = mg$ is constrained to move in a vertical circle of radius l under the uniform gravitational force after having been set in motion and released. If the particle obeys the laws of quantum mechanics, is the angular momentum conserved? Prove your statement.

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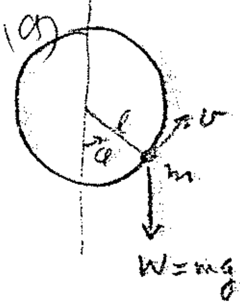
a) (10)



In order for decide whether L_z is conserved, we must calculate $\frac{d}{dt}\langle L_z \rangle$.

Since for any operator $\hat{F}$ which appears in H $\quad \frac{d}{dt}\langle \hat{F} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{F}] \rangle$

we must inquire whether $[H, F] = 0$ for an arbitrary state to have $\frac{d}{dt}\langle F \rangle =$

now since $H = \frac{p_\varphi^2}{2I} + V = \frac{p_\varphi^2}{2I} + mgl(1 - \cos\varphi)$ and $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$

it is immediately obvious that $V L_z - L_z V \neq 0$

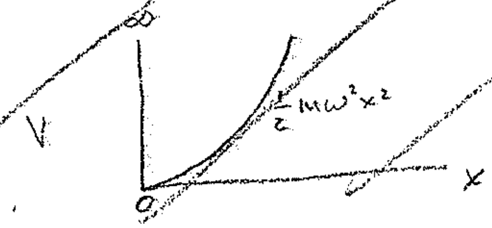
so that $[H, L_z] \neq 0$

and hence $\frac{d}{dt}\langle L_z \rangle \neq 0$ angular momentum is not conserved.

QM (U)

a) What are the energy levels of a particle moving in the one-dimensional potential

$$V(x) = \begin{cases} \infty & x<0 \\ \frac{1}{2}m\omega^2x^2 & x>0 \end{cases}$$



b) Using the Heisenberg uncertainty relation $pr \sim \hbar$ for the "best" localization of a particle in orbit, and a variational principle, estimate the binding energy (in eV) of a K-shell electron in carbon. Neglect relativistic effects.

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a) We require that $\psi \to 0$ at $x=0$, $\psi = \psi_{SHO}$ $x>0$

These conditions are just those for the odd solutions of the linear oscillator.

thus, $E_n = (n+\frac{1}{2})\hbar\omega$, n odd

or, writing $n = 2k+1$ $\quad \underline{E_k = (2k+3/2)\hbar\omega \quad (k=0,1,2,\ldots)}$.

12 b) For a K-shell electron, we can neglect screening of the nuclear coulomb field by other electrons; the energy is then, using $p^2 \sim \frac{\hbar^2}{r^2}$

$$E = \frac{p^2}{2m} - \frac{Ze^2}{r} \sim \frac{\hbar^2}{2mr^2} - \frac{Ze^2}{r}$$. This is a minimum for

$$\frac{\partial E}{\partial r} = -\frac{\hbar^2}{mr^3} + \frac{Ze^2}{r^2} = 0, \quad \text{so } r = \frac{\hbar^2}{Zme^2} = \frac{a_0}{Z}$$

then $$E = \frac{\cancel{\hbar^2} Z^2m^2e^4}{2\cancel{m}\hbar^{4}{}^{2}} - \frac{Z^2me^4}{\hbar^2} = -Z^2\frac{me^4}{2\hbar^2} = -Z^2(13.6\,eV) = -36(13.6)$$

$$\underline{= -489.6\ eV}$$

$-R\,Z^2$