#14

Winter 1973

This Comprehensive Examination for Winter 1973 consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half are graduate level. Work carefully, and show all your steps, so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the blue-book — you may get some credit for it!

If something is omitted from the statement of the problem or you feel there are ambiguities, get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books and papers, except pencil and blue-book, on the floor.

Some information you may find useful follows:

$$\pi = 3.14159$$

$$(1+\delta)^n \approx 1+n\delta+ \dots \quad \delta \ll 1$$

$$\rho e^{i\phi} = x+iy$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \tan^{-1} x \quad , \qquad \int \frac{y^3\,dy}{z^{3/2}} = z^{\frac{1}{2}} + \frac{x^2}{z^{\frac{1}{2}}}$$

where $z = x^2 + y^2$

$$\int \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The first few eigenfunctions for the one-dimensional simple harmonic oscillator of mass m and frequency ω are

$$\psi_0 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{X^2}{2}} \; , \quad \psi_1 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sqrt{2}\, X e^{-\frac{X^2}{2}} \; , \quad \psi_2 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sqrt{2}\,(X^2-\tfrac{1}{2}) e^{-\frac{X^2}{2}}$$

where

$$X = \sqrt{\frac{m\omega}{\hbar}}\; x$$

(1) Non-relativistic quantum mechanics is based on the postulate that the probability amplitude ψ of a system obeys the Schrödinger equation

$\hat{H}\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}$ where $\hat{H}$ is the Hamiltonian operator. Further, the probability amplitude describes a state in which there is <u>no</u> uncertainty in the value of any dynamical quantity f if ψ obeys the eigenvalue equation $\hat{F}\psi = F_0\psi$ where $\hat{F}$ is the operator corresponding to the quantity whose precise value (eigenvalue) is F_0.

Consider a particle of mass m in one dimension bound in the potential $V(x) = \frac{1}{2}kx^2$. Working in coordinate representation,

a) Write the Hamiltonian function for the system.

b) Write the Hamiltonian operator for the system.

c) Write the equation of motion for $\psi(x,t)$.

d) Assuming the system has eigenvalues E_n with corresponding ψ_n, derive the explicit time dependence of the $\psi_n(x,t)$. Define or calculate all terms.

e) Given the set of $\psi_n(x,t)$, what is the probability amplitude for <u>any</u> state of the system? Define or calculate all terms.

f) If the potential is now changed to $V = \frac{1}{2}kx^2 + bx^3$, where the second term is small compared to the first, calculate the first-order correction to the second energy eigenvalue of the system. Show clearly what you are doing and why.
(Hint: An exact knowledge of $\psi_2(x,t)$ is not required.)

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(2) A thin plastic circular disk of radius R has charge q uniformly distributed over its surface. The disk is rotating with angular frequency ω about its axis.

a) Compute the magnetic field along the axis.

b) Calculate the magnetic dipole moment of the disk.

c) A second identical disk with the same $\vec{\omega}$ as the first is placed on the axis a distance L away from the first disk. Assume that $L \gg R$. Calculate the energy of interaction of the two disks.

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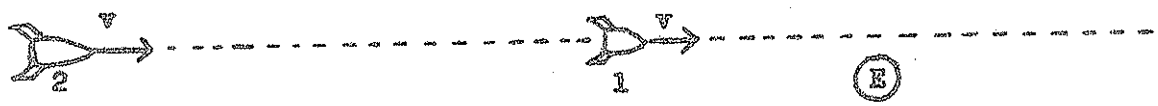
(3)



Two space ships are traveling along the same straight line with constant velocity. Observers on earth measure the velocity of each to be $v = 0.8\ c$ as it passes earth, and note that ship 2 passes earth exactly one year after ship 1. At the instant it passes earth, ship 2 transmits a radio message to ship 1. Immediately upon receiving the message, ship 1 transmits a reply to ship 2, and to earth.

a) How long (by its own clock) after it passes earth does ship 1 receive the message?

b) How long (by its own clock) after it sent the message does ship 2 receive the reply?

c) How long (by earth clocks) after ship 2 passes earth does the reply reach earth?

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(4) A pair of circular metal plates of radius R and separated by a small distance h in vacuum, is connected to a generator of frequency ω. This arrangement might produce a uniform electric field between the plates $E = E_0 e^{i\omega t}$. Of course this is nonsense - it is exactly correct only on the axis of the system. Write Maxwell's equations for the electromagnetic field at all points between the plates and neglecting edge effects use an iterative method to calculate corrections to the electric field within the capacitor at arbitrary radius r from the center, to fourth order in r.

(Continue to next page.)

(5)

a) If a high-energy particle scatters off some target particle, the square of the invariant momentum transfer to the incident particle is defined to be:

$$t = (\underline{p}_f - \underline{p}_i)^2$$

where $\underline{p}_i$, $\underline{p}_f$ are the 4-vector momenta of the incident and scattered particles. $(\underline{v})^2$ is the square of the invariant length of $\underline{v}$.

A high energy π^+ beam is sent into a liquid hydrogen bubble chamber to study the reaction $\pi^+ + p \rightarrow \pi^+ + p$. The kinetic energy, T_p of the final state proton is determined for each such interaction by measuring its range in the hydrogen.

Show that for each event

$$t = -2MT_p \qquad (M = \text{proton mass}) .$$

b) The same definition for t can be used for the reaction $\pi^- + p \rightarrow \pi^- + N^*$, where N^* is an excited state of the nucleon. Let the energy of excitation of this state be Δ. Suppose that the kinetic energy, T_{N^*}, of the final state N^* is measured in the bubble chamber.

Derive an expression for t in terms of M, Δ, T_{N^*} .

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(6) Consider a system of N non-interacting atoms (for example, a very dilute gas) with spin $\frac{1}{2}\hbar$ and dipole moment μ_0 at an absolute temperature T in a strong magnetic field B.

a) Compute the absolute probability of finding an atom with its magnetic moment aligned with the field.

b) Calculate the mean magnetic moment and investigate its value for $\mu_0 B \ll kT$ and $\mu_0 B \gg kT$, respectively.

c) Calculate the temperature dependence of the heat capacity of the system for high temperatures ($\mu_0 B \ll kT$).

Solutions - Comp #14

#1) a) $H(x, p_x) = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = T + V$ $\qquad \hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$ 2 points

b) $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2$ 2 points

c) $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}kx^2\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}$ 2 points

d) $\hat{H}\psi_n = -\frac{\hbar}{i}\frac{\partial\psi_n}{\partial t} = E_n\psi_n$ so $\frac{\partial\psi_n(x,t)}{\psi_n(x,t)} = -\frac{i}{\hbar}E_n t$ 4 points

and, at most $\underline{\psi_n(x,t) = u_n(x)\, e^{-\frac{i}{\hbar}E_n t}}$ where $\hat{H}u_n(x) = E_n u_n(x)$

so that $-\frac{\hbar^2}{2m}\frac{d^2u_n}{dx^2} + \left(\frac{1}{2}kx^2 - E_n\right)u_n = 0$ defines u_n

e) Since the $u_n(x)$ are an orthonormal set, $\underline{\psi(x,t) = \sum_{n=0}^{\infty} A_n u_n(x)\, e^{-\frac{i}{\hbar}E_n t}}$ 5 points

then, since $\psi(x,0) = \sum A_n u_n$, we can multiply by u_m^* and integrate

$$\langle u_m | \sum A_n u_n \rangle = \langle u_m | \psi(x,0) \rangle$$

$$A_m \langle u_m | u_m \rangle + \underbrace{\langle \text{orthogonal terms} \rangle}_{\to 0} = \langle u_m | \psi(x,0) \rangle$$

or $\underline{A_m = \langle u_m | \psi(x,0) \rangle}$

5 points

f) Since the second term is small, we can regard bx^3 as a perturbation; the first order correction to the energy of <u>any</u> eigenstate is then

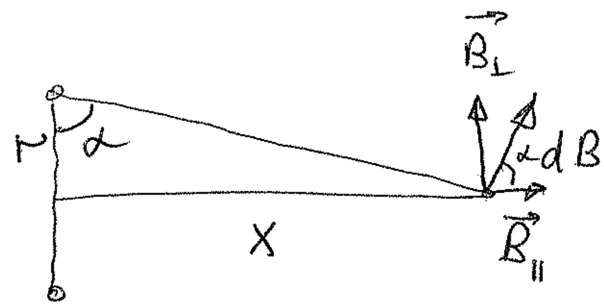
$$E^{(1)} = H^{(1)}_{nn} = \langle \psi_n^{(0)} | bx^3 | \psi_n^{(0)} \rangle = 0$$

This integral is zero even without performing the integration; since each state has definite parity, then for an odd perturbation like bx^3, the integrand is odd and the integral over symmetric limits ($\pm\infty$) will vanish for all eigenstates, and there is <u>zero</u> first-order corrections

② Solution E+m undergrad comp exam.

a) B along axis due to a single current loop (i)

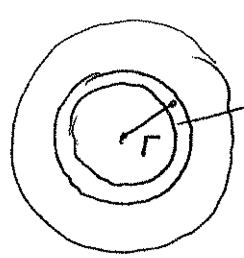
$$d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{\vec{dl} \times \vec{h}}{h^3} \quad \text{(Biot - Savart law)}$$



$$B_\perp = 0 \qquad d\vec{B}_\parallel = \frac{\mu_0 i}{4\pi} \frac{dl}{h^2} \cos\alpha$$

$$= \frac{\mu_0 i}{4\pi} \frac{r}{(x^2+r^2)^{3/2}} dl$$

$$B_\parallel = \frac{\mu_0 i}{2} \frac{r^2}{(x^2+r^2)^{3/2}}.$$



$$dq = \frac{2\pi r \, dr}{\pi R^2} q$$

$$di = \frac{dq}{T} = \frac{\omega r \, dr}{\pi R^2} q$$

E+M undergrad solution

$$dB_{\parallel} = \frac{\mu_0}{2\pi}\frac{\omega q}{R^2}\frac{r^3\,dr}{(x^2+r^2)^{3/2}}$$

$$\int \frac{y^3 dy}{z^{3/2}} = z^{1/2} + \frac{x^2}{z^{1/2}}$$

$$z = x^2 + y^2$$

$$B_{\parallel} = \frac{\mu_0}{2\pi}\frac{\omega q}{R^2}\left[(x^2+R^2)^{1/2} + \frac{x^2}{(x^2+R^2)^{1/2}} - 2x\right]$$

b). dipole moment of disk.

$$d\mu = di\,A \qquad di = \frac{\omega q}{\pi R^2}\,r\,dr \qquad A = \pi r^2.$$

$$\mu = \int_0^R \left(\frac{\omega q}{R^2}\right) r^3 dr = \frac{\omega q R^2}{4}.$$

c) interaction energy

$$\mathcal{E} = -\mu \cdot B + \frac{q^2}{4\pi\varepsilon_0 L^2}$$

$$= -\frac{\mu_0 \omega^2 q^2}{8\pi}\left[(L^2+R^2)^{1/2} + \frac{L^2}{(L^2+R^2)^{1/2}} - 2L\right] \quad L \gg R$$

$$\mathcal{E} = -\frac{\mu_0 \omega^2 q^2}{8\pi}\left[L + \frac{1}{2}\frac{R^2}{L} + L^2\left(\frac{1}{L}\left(1 - \frac{1}{2}\frac{R^2}{L}\right)\right) - 2L\right]$$

$$\underbrace{\left[L + \frac{1}{2}\frac{R^2}{L} + L - \frac{1}{2}\frac{R^2}{L} - 2L\right]}_{0}$$

must consider next order.

$$\left[L\left(1 + \frac{1}{2}\frac{R^2}{L^2} - \frac{1}{8}\frac{R^4}{L^4}\right) + L\left(1 - \frac{1}{2}\frac{R^2}{L^2} + \frac{3}{8}\frac{R^4}{L^4}\right)\right]$$

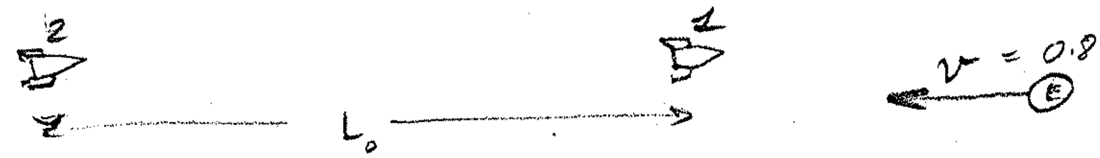
$$\left[\frac{1}{4}\frac{R^4}{L^3}\right]$$

$$\underline{\underline{\mathcal{E} = -\frac{\mu_0 \omega^2 q^2 R^4}{32\pi}\frac{1}{L^3}}} + \frac{q^2}{4\pi\varepsilon_0 L^2}$$

Underlined Rel.

(3)

It is easiest to work the problem in the system in which the space ships are at rest:



In the earth frame the distance between ships is $L = vt = .8$ l.y

thus $L_o = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}}$ $\frac{.8}{\sqrt{1-.8^2}} = \frac{.8}{.6} = 1.33$ l.yr.

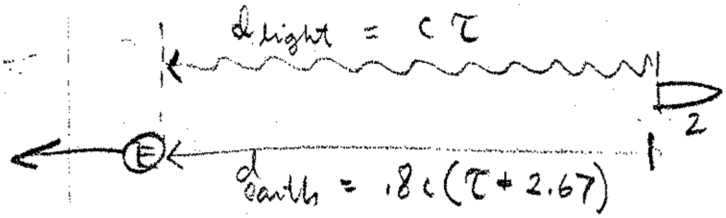
So the time for a radio message to pass between the ships is $t = \frac{L_o}{c} = 1.33$ years

The delay between the earth passing ship 1 and the sending of the message is the time it takes the earth to move from ship 1 to ship 2 : $t = L_o/.8c = 1.33/.8 = 1.67$ yr

(a) $t = t(\text{earth } 1 \to 2) + t(\text{light } 2 \to 1) = 1.67 + 1.33 = \underline{3.0 \text{ yr}}$

(b) $t = t(\text{light } 2 \to 1) + t(\text{light } 1 \to 2) = 1.33 + 1.33 = \underline{2.67 \text{ yr}}$

(c) Earth will receive the message a time τ later than ship 2 because it has passed beyond ship 2:



$d_{light} : d_{earth}$

$c\tau = .8c(\tau + 2.67)$

$\tau(1-.8) = .8(2.67)$

$\tau = \frac{.8 \times 2.67}{.2} = 10.67$

thus $t = 10.67 + 2.67 = 13.33$ yr

However, the time according to earth observers will be reduced by the Lorentz factor:

$t_o = t\sqrt{1-\frac{v^2}{c^2}} = 13.33 \times .6 = \underline{8 \text{ yr.}}$

For the vacuum between the plates, Maxwell's equations are

#4) $\nabla\cdot\vec{D}=0$

$\nabla\cdot\vec{B}=0$

$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

$\nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}$

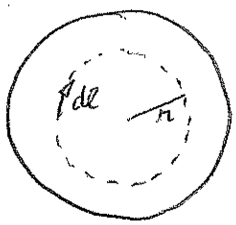
The last two equations are easiest applied in integral form in the present problem:

$$\oint\vec{E}\cdot d\vec{l}=-\frac{\partial}{\partial t}\int\vec{B}\cdot d\vec{S} \quad (1)$$

$$\oint\vec{H}\cdot d\vec{l}=\frac{\partial}{\partial t}\int\vec{D}\cdot d\vec{S}$$ but $\vec{B}=\mu_0\vec{H}$ and $\vec{D}=\varepsilon_0\vec{E}$ in the vacuum.

or $\oint\vec{B}\cdot d\vec{l}=\mu_0\varepsilon_0\frac{\partial}{\partial t}\int\vec{E}\cdot d\vec{S}$ and finally, $\oint\vec{B}\cdot d\vec{l}=\frac{1}{c^2}\frac{\partial}{\partial t}\int\vec{E}\cdot d\vec{S}$ (2)

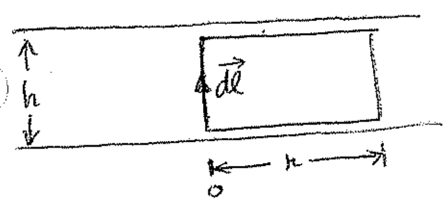
In zeroth order, the electric field is everywhere uniform and is $E_1=E_0e^{i\omega t}$. However, since $E_1=E_1(t)$, at radius r there is a B_1 found from (2) and the following geometry:



$$B_1\cdot 2\pi r=\frac{i\omega E_1}{c^2}\pi r^2$$

so $B_1=\frac{i\omega r}{2c^2}E_1=B_1(t)$

This changing magnetic induction field will produce an E_2, which we find by applying (1) to the following geometry, remembering that $E_2(0)=0$ and that the electric field is vertical



$$-E_2h=-\frac{\partial}{\partial t}\int\vec{B}_1\cdot d\vec{S}=\frac{\omega^2}{2c^2}\int_0^r rE_1\,h\,dr$$

or $E_2=-\frac{\omega^2r^2}{4c^2}E_1$

This will contribute more $B_2\cdot 2\pi r=\frac{1}{c^2}\frac{\partial}{\partial t}\int E_2\,2\pi r\,dr=\frac{i\omega}{c^2}\left(-\frac{\omega^2}{4c^2}\right)\frac{r^4}{4}E_1$

So $B_2=-\frac{i\omega^3}{16c^4}r^3E_1$

giving, in turn, $-E_3h=-\frac{\partial}{\partial t}\int_0^r-\frac{i\omega^3}{16c^4}r^3E_1\,h\,dr$

or $E_3=+\frac{\omega^4r^4}{64c^4}E_1$

So that $$E=E_1\left(1-\frac{1}{4}\frac{\omega^2r^2}{c^2}+\frac{1}{64}\frac{\omega^4r^4}{c^4}\right)$$

(5) Grad Rel.

$$\pi^+ + p \longrightarrow \pi^+ + p$$

4-momenta: $\underline{p_i} + \underline{q_i} = \underline{p_f} + \underline{q_f}$ (conservation of energy, momentum)

thus $(\underline{p_f} - \underline{p_i}) = (\underline{q_i} - \underline{q_f})$

$$q_i = \left(\frac{E_i}{c}, 0\right) = \left(\frac{Mc^2}{c}, 0\right) = (Mc, 0)$$

$$q_f = \left(\frac{E_f}{c}, \vec{q}_f\right)$$

$$q_i - q_f = \left(Mc - \frac{E_f}{c}, \vec{q}_f\right)$$

$$t = (q_i - q_f)^2 = \left(Mc - \frac{E_f}{c}\right)^2 - |\vec{q}_f|^2 = \frac{E_f^2}{c^2} - |\vec{q}_f|^2 + M^2c^2 - 2ME_f$$

but $E_f^2 = M^2c^4 + |\vec{q}_f|^2c^2 \Rightarrow \frac{E_f^2}{c^2} - |\vec{q}_f|^2 = M^2c^2$

so $t = 2M^2c^2 - 2ME_f = -2M(E_f - Mc^2) = \underline{\underline{-2MT_p}}$

(b) Same as above, but (letting $M_{N^*} = M + \frac{\Delta}{c^2}$)

$$E_f^2 = M_{N^*}^2c^4 + |\vec{q}_f|^2c^2 \Rightarrow \frac{E_f^2}{c^2} - |\vec{q}_f|^2 = M_{N^*}^2c^2$$

so $t = M_{N^*}^2c^2 + M^2c^2 - 2ME_f = (M_{N^*}^2 + M^2)c^2 - 2M(M_{N^*}c^2 + T_{N^*})$

$$= (M_{N^*}^2 - 2MM_{N^*} + M^2)c^2 - 2MT_{N^*}$$

$t = (M_{N^*} - M)^2c^2 - 2MT_{N^*}$ but $M_{N^*} - M = \frac{\Delta}{c^2}$

$$t = \underline{\underline{\frac{\Delta^2}{c^2} - 2MT_{N^*}}}$$

Stat Mech. Comp Exam Solution.

(6)

a). interaction energy of atom with μ_0 along B

$$\mathcal{E}_+ = -\mu_0 B$$

interaction energy of atom with μ_0 opposite B

$$\mathcal{E}_- = \mu_0 B$$

Canonical distribution function $B\uparrow \uparrow \mu_0$

$$P_+ = C e^{-\frac{\mu_0 B}{kT}}$$

for $B \uparrow\downarrow$.

$$P_- = C e^{\frac{\mu_0 B}{kT}}$$

$$P_+ + P_- = C\left(e^{\frac{\mu_0 B}{kT}} + e^{-\frac{\mu_0 B}{kT}}\right) = 1 \quad \text{determines } C.$$

Thus.

$$P_+ = \frac{e^{-\frac{\mu_0 B}{kT}}}{e^{\frac{\mu_0 B}{kT}} + e^{-\frac{\mu_0 B}{kT}}}.$$

b). $$\bar{\mu} = (P_+ - P_-)\mu_0 = \frac{e^{\frac{\mu_0 B}{kT}} - e^{-\frac{\mu_0 B}{kT}}}{e^{\frac{\mu_0 B}{kT}} + e^{-\frac{\mu_0 B}{kT}}}\mu_0$$

Stat Mech Solution. ②

$$\bar{\mu} = \mu_0 \tanh\left(\frac{\mu_0 B}{kT}\right)$$

for $\mu_0 B \ll kT$ $\quad e^{\frac{\mu_0 B}{kT}} \simeq 1 + \frac{\mu_0 B}{kT}$ etc

$$\bar{\mu} = \mu_0\left(\frac{\mu_0 B}{kT}\right) = \frac{\mu_0^2 B}{kT}$$

for $\mu_0 B \gg kT$ $\quad e^{\frac{\mu_0 B}{kT}} \gg e^{-\frac{\mu_0 B}{kT}}$.

$$\bar{\mu} = \mu_0$$

c) The energy of the system is

$$\bar{E} = -\mu_0 B N (P_+ - P_-)$$

$$= -\mu_0 B N \left[\frac{e^{-\frac{\mu_0 B}{kT}} - e^{\frac{\mu_0 B}{kT}}}{e^{-\frac{\mu_0 B}{kT}} + e^{\frac{\mu_0 B}{kT}}} \right]$$

heat cap

$$C = \frac{d\bar{E}}{dT} = \ldots\ldots$$