No. 13

Fall 1972

This Comprehensive Examination for Fall 1972 consists of six problems of equal weight (20 points each). Half the problems are judged to be at intermediate undergraduate level, the other half are graduate level. Work carefully, and show all your steps, so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the blue-book -- you may get some credit for it!

If something is omitted from the statement of the problem or you feel there are ambiguities, get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books and papers, except pencil and blue-book, on the floor.

Some information you may find useful follows:

$$\pi = 3.14159$$

$$(1+\delta)^n \approx 1+n\delta+ \ldots\ldots \quad \delta<<1$$

$$\rho e^{i\phi} = x+iy$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \tan^{-1}x$$

$$\int \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2}$$

$$\sigma_x = \begin{pmatrix} o & 1 \\ 1 & o \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} o & -i \\ i & o \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & o \\ o & -1 \end{pmatrix}$$

The first few eigenfunctions for the one-dimensional simple harmonic oscillator of mass m and frequency ω are

$$\psi_o = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{X^2}{2}}, \quad \psi_1 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sqrt{2}\, Xe^{-\frac{X^2}{2}}, \quad \psi_2 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sqrt{2}\,(X^2-\tfrac{1}{2})e^{-\frac{X^2}{2}}$$

where

$$X = \sqrt{\frac{m\omega}{\hbar}}\; x$$

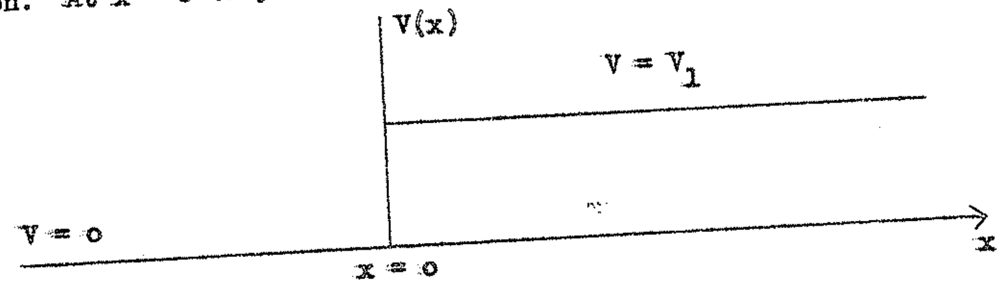
No. 13

COMPREHENSIVE EXAMINATION

1. a) Explain briefly how a quarter-wave plate is made and how it works. Describe how a quarter-wave plate and a linear polarizer can be used to determine the state of polarization of arbitrarily polarized light.

   b) Light traveling in the +z direction passes first through a quarter-wave plate whose fast axis is along the x-axis, and then through a linear polarizer whose transmission axis is $+30^o$ from the x-axis. It is found that the intensity of the transmitted light is zero. Describe completely (and justify your statements) the state of polarization of the incident light.

2. A stream of particles of kinetic energy E is moving in the positive x-direction. At x = o they hit a potential step (see Fig.).



   a) Derive the wave functions in the regions $x<o$ and $x>o$ for the case when the particle energy is greater than V_1.

   b) Determine the reflection (R) and transmission (T) coefficients. (R is the ratio of the reflected current to the incident current, and T is the ratio of the transmitted current to the incident current.)

   c) For the case $E<V_1$ what are the values of R and T? State your conclusions and briefly sketch the proof without detailed calculations.

3. A particle moves from rest at the origin at time t = o in an inertial frame of reference under the influence of a constant fare $\vec{F}=mg\hat{i}$ where $\hat{i}$ is the unit vector in the x-direction. In this problem it is not necessary that $v<<c$.

   a) Calculate the velocity and position of the particle as a function of time.

   b) Draw graphs of v(t) and x(t) and compare these curves with the non-relativistic results.

4. A two-dimensional ~~linear~~ simple harmonic oscillator is in an angular momentum eigenstate with eigenvalue $L_z = -\hbar$. Write the normalized eigenfunction, and calculate the eigen-energy of the state. Justify all steps if you want to get credit.

COMPREHENSIVE EXAMINATION

5. Two particles of spin ½ and magnetic moments μ_1, μ_2 ($\mu_1 > \mu_2 > 0$) interact with each other magnetically, and also with an external magnetic induction field $\vec{B}$. The Hamiltonian is:

$$\mathcal{H} = -(\mu_1 \vec{\sigma}_1 + \mu_2 \vec{\sigma}_2)\cdot\vec{B} + c\,\vec{\sigma}_1\cdot\vec{\sigma}_2 \qquad c = \text{constant}$$

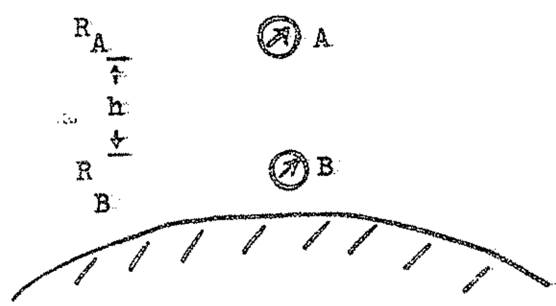
   a) Take $\vec{B}$ to be in the +z- direction. Write the Hamiltonian matrix explicitly in representation that has as basis states: $(\uparrow\uparrow)$, $(\uparrow\downarrow)$, $(\downarrow\uparrow)$, $(\downarrow\downarrow)$. (Each particle in an eigenstate of σ_z.)

   b) Derive exact expressions for the energies of the eigenstates of the system.

   c) Give approximate expressions for the energy levels valid for (i) small B (first order in B) and (ii) very large B.

   d) Draw a rough graph showing the energy levels as a function of B.

6. Einstein's Principle of Equivalence says it is impossible to distinguish between the effects of a uniform acceleration and a uniform gravitational field. Two clocks, A and B, having been synchronized in free space at relative rest, are placed at the top and bottom of a 100 meter high building. Using the Principle of Equivalence, calculate which one runs slower. By how much

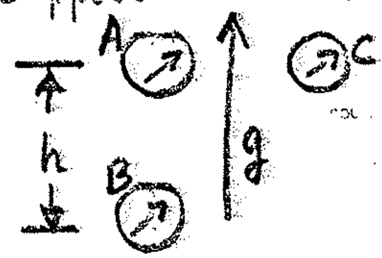


will a time interval measured on the ground (by B) as one second, differ from that measured by A?

(In your calculations, assume that any v<<c so that only terms of lowest order in v/c need be kept.)

A uniform gravitational field can exist only over limited regions of space (otherwise bodies could be accelerated indefinitely to acquire infinite momentum). Let us assume that any velocities that can be acquired are small, $v \ll c$.

Consider two clocks A and B separated by a distance h and subject to a uniform acceleration g. Let both clocks sweep by a third clock C in free space (suppose all three clocks were synchronized in free space at rest



Suppose the clocks have velocities v_A and v_B as they pass successively the clock C ($v_A \neq v_B$ since B accelerated through an extra distance h). We now apply Special Relativity to find the relation between a time interval ΔT_C of fixed duration as measured by C and the corresponding intervals ΔT_A and ΔT_B ~~as measured by the clocks A and B as they each pass C in turn.~~

Thus $\Delta T_A = \Delta T_C \left(1 - \frac{v_A^2}{c^2}\right)^{-1/2} \approx \Delta T_C \left(1 + \frac{1}{2}\frac{v_A^2}{c^2}\right)$ we drop all but terms of order $\frac{v^2}{c^2}$ since $v \ll c$

and $\Delta T_B = \Delta T_C \left(1 - \frac{v_B^2}{c^2}\right)^{-1/2} \approx \Delta T_C \left(1 + \frac{1}{2}\frac{v_B^2}{c^2}\right)$

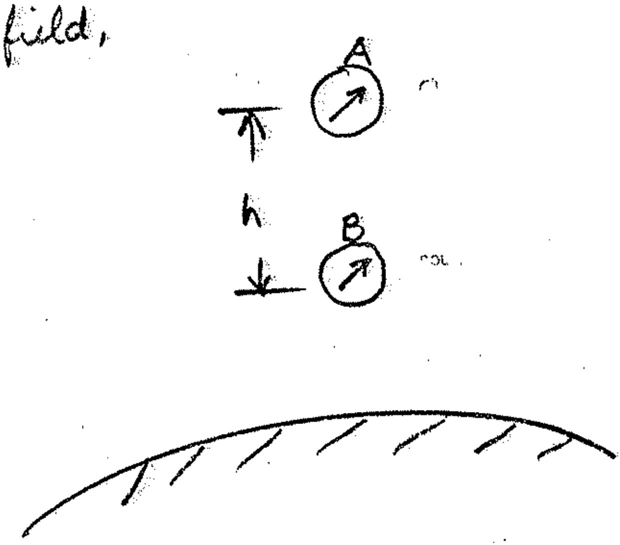
so $\Delta T_B = \Delta T_A \left(1 + \frac{1}{2}\frac{v_A^2}{c^2}\right)^{-1}\left(1 + \frac{1}{2}\frac{v_B^2}{c^2}\right) \approx \Delta T_A \left(1 - \frac{1}{2}\frac{v_A^2}{c^2}\right)\left(1 + \frac{1}{2}\frac{v_B^2}{c^2}\right) \approx \Delta T_A \left(1 + \frac{1}{2}\frac{v_B^2 - v_A^2}{c^2}\right)$

but since both clocks are accelerating uniformly at g, $v_B^2 = v_A^2 + 2gh$

So that $\Delta T_B \approx \Delta T_A \left(1 + \frac{gh}{c^2}\right)$

Note that mgh has the dimensions of potential energy, so gh has the dimensions of potential.

ow, since the Principle of Equivalence states that it is impossible to distinguish between the effects of a uniform acceleration and a uniform gravitational field, the fact that identical clocks which keep different time under uniform acceleration (as we have seen) implies that those same clocks will also keep different time when at rest, but separated by a distance h in a uniform gravitational field,



In fact, the requirement must be

$$\Delta T_B = \Delta T_A \left(1 + \frac{gh}{c^2}\right) = \Delta T_A \left(1 + \frac{U_B - U_A}{c^2}\right)$$

where U_A and U_B are the gravitational potentials at A and B, respectively.

Thus
$$\Delta T_A \approx \Delta T_B \left(1 + \frac{U_A - U_B}{c^2}\right) = \Delta T_B \left[1 - GM\left(\frac{1}{R_A} - \frac{1}{R_B}\right)\frac{1}{c^2}\right]$$
$$\approx \Delta T_B \left[1 - GM\left(\frac{R_B - R_A}{R^2}\right)\frac{1}{c^2}\right]$$
$$= \Delta T_B \left[1 - \frac{GM}{R^2}\frac{(R_B - R_A)}{c^2}\right]$$

Now, for a body at rest at the earth's surface, we know that $\cancel{m}g = \frac{GM\cancel{m}}{R^2}$
and since $R_B - R_A = -h$, $\Delta T_A \approx \Delta T_B \left(1 + \frac{gh}{c^2}\right)$ in agreement with the Principle of Equivalence, we get the same result as before.

$g \approx 10 \frac{m}{sec^2}$ so $\frac{gh}{c^2} \approx 10^{-14}$
$h \approx 100$ m

for $\Delta T_B = 1$ sec, $\Delta T_A > \Delta T_B$ by one part in 10^{14}; the upper clock runs slow.

Undergrad Optics No. 13

a). Explain briefly how a quarter-wave plate is made, and how it works and a Describe linear polarizer can be used to determine the state of polarization of arbitrarily polarized light.

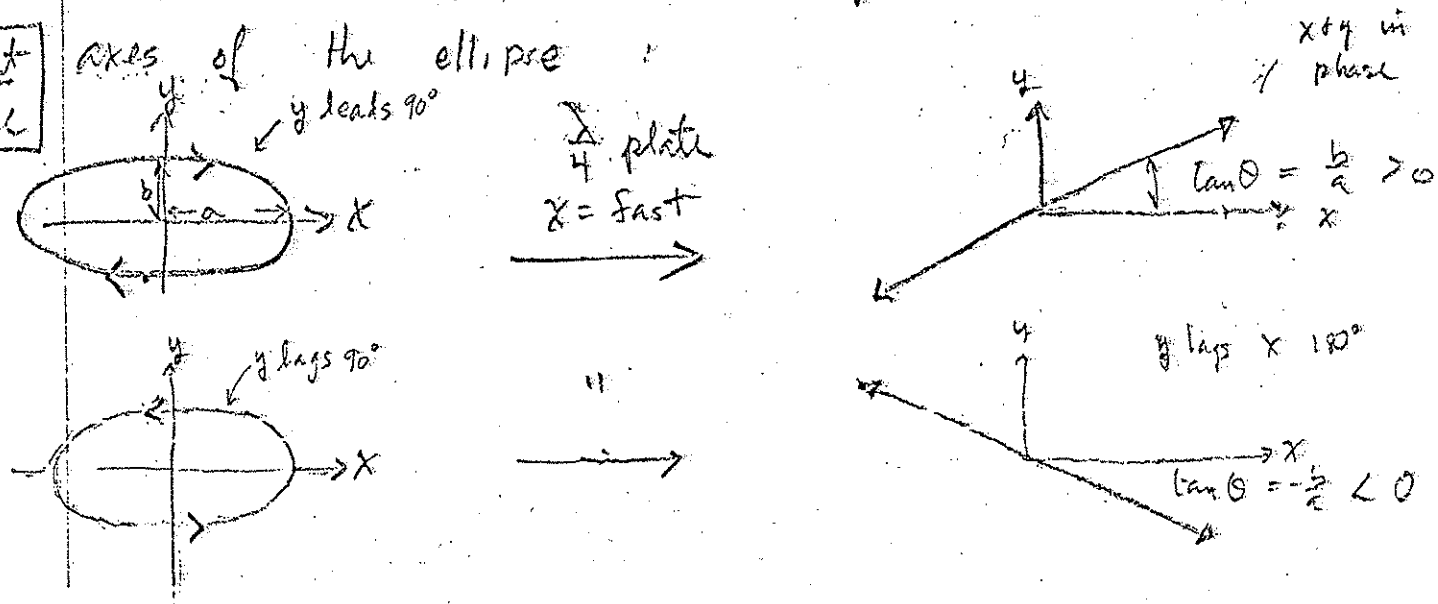
b) Light travelling in the +z direction first passes through a quarter-wave plate whose fast axis is along the x-axis, and then through a linear polarizer whose transmission axis is +30° from the x axis. It is found that ~~the~~ the intensity of the transmitted light is zero. Describe completely the state of polarization of the incident light.

A quarter-wave-plate whose fast axis is in the x-direction retards the phase of the y component 90° with respect to the x component. Thus elliptically polarized light whose axes are aligned along the x-y axes will come out linearly polarized with x + y components equal to the x + y axes of the ellipse.

light out of page

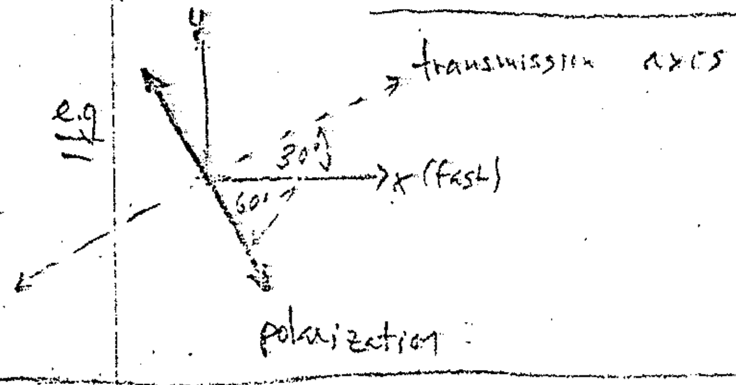
Right Hand Elliptical

Left Hand Elliptical

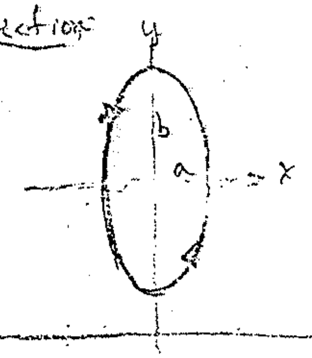


Thus to determine polarization, the $\lambda/4$ plate + polarizer are adjusted to minimize* transmitted intensity. θ, the angle between the fast axis and the polarization into the polarizer gives $b/a = \tan\theta$. If $\theta > 0$, then RH; if $\theta < 0$, LH.

e.g.



Polarization



Left Hand Elliptical

$\frac{b}{a} = \tan 60 = \sqrt{3}$

* The residual intensity is $\frac{1}{2}$ the unpolarized component

QM question Undergrad

answer.

a)

$$\psi_{x<0} = \left(Ae^{ik_0x} + Be^{-ik_0x}\right)e^{-i\omega t}$$

$$\omega = \frac{E}{\hbar} \qquad k_0^2 = \frac{2mE}{\hbar^2}$$

$$\psi_{x>0} = Ce^{ik_1x}e^{-i\omega t}$$

$$k_1^2 = \frac{2m(E-V_1)}{\hbar^2}$$

b) ψ and $\frac{d\psi}{dx}$ must be continuous at $x=0$

$$A+B = C \qquad k_0(A-B) = k_1 C$$

Current densities are $j_B = -\rho_B\left(\frac{\hbar k_0}{m}\right)$

$$j_C = \rho_C\left(\frac{\hbar k_1}{m}\right)$$

$$\rho_B = |B|^2 \quad \rho_C = |C|^2.$$

$$\frac{B}{A} = \frac{k_0-k_1}{k_0+k_1}$$

$$R = \left|\frac{j_B}{j_A}\right| = \left(\frac{k_0-k_1}{k_0+k_1}\right)^2$$

QM question

answer cont:

$$\frac{C}{A} = \frac{2k_0}{k_1+k_0} \quad \therefore \quad T = \left|\frac{j_C}{j_A}\right| = 4\frac{k_1}{k_0}\frac{k_0^2}{(k_1+k_0)^2} = 4\frac{k_1 k_0}{(k_1+k_0)}$$

Note that $R+T=1$

c) for $E < V_1$ $\quad k_1 = i\mu \quad \mu = \text{real}.$

thus $$\frac{B}{A} = \frac{1-\frac{i\mu}{k_0}}{1+\frac{i\mu}{k_0}} \qquad \mu = \sqrt{\frac{2m(V_1-E)}{\hbar^2}}$$

$$\left|\frac{j_B}{j_A}\right| = \left|\frac{B}{A}\right|^2 = 1 \qquad \underline{R=1 \quad T=0}$$

Relativity Question

answer

a) eq of motion is

$$m \frac{d}{dt}\left[\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right] = mg$$

boundary cond $v=0$ at $t=0$

integration yields

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = gt$$

$$\therefore \quad v = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$$

Note that $v \le c$ for all values of t

and $v \to c$ as $t \to \infty$

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}} \qquad X = \int_0^t \frac{gt'\,dt'}{\sqrt{1+\frac{g^2t'^2}{c^2}}}$$

(hint) useful steps

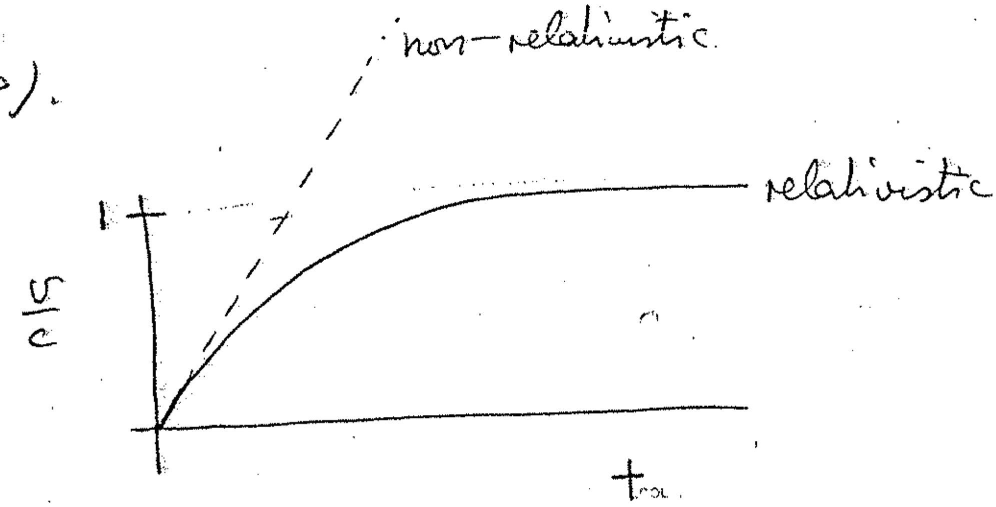
give integral

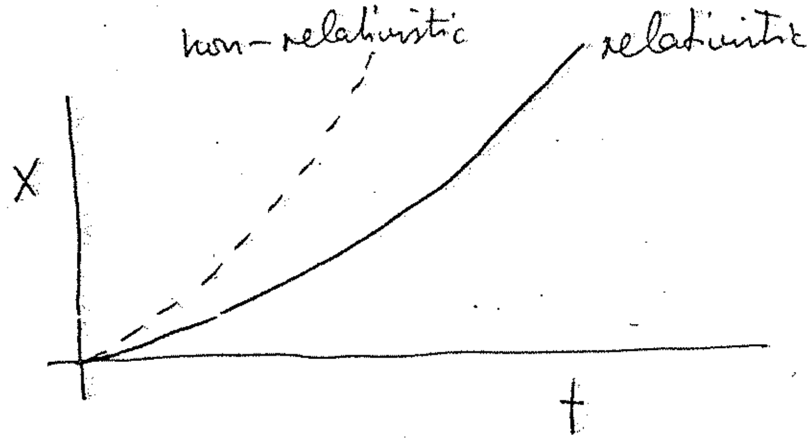
$$\int \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2}$$

$$X = \frac{c^2}{g}\left[\sqrt{1+\frac{g^2t^2}{c^2}} - 1\right]$$

Relativity question
answer cont.

b).





7. A two-dimensional linear harmonic oscillator is in an eigenstate of angular momentum with eigenvalue $L_z = -\hbar$. Write the normalized eigenfunction, and calculate the eigen-energy of the state.

~~Note: The first few eigenfunctions for the one-dimensional simple harmonic oscillator of mass m and frequency $\omega = \sqrt{\frac{k}{m}}$ are~~

$$\psi_0 = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{X^2}{2}}$$

$$\psi_1 = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \sqrt{2}\, X e^{-X^2/2} \qquad X = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\psi_2 = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \sqrt{2}\left(X^2 - \frac{1}{2}\right) e^{-X^2/2}$$

Sol'n: Given the one-dimensional eigenfunctions for the SHO, the two dimensional eigenfunctions are ~~easily constructed~~ as products of the ψ_n's, since the S-equation is separable

Thus, $\Psi_n(x,y) = \psi_{n_x}(X)\,\psi_{n_y}(Y)$, $\quad E_n = (n_x + n_y + 1)\hbar\omega$

Thus, the lowest energy level, E_0, is given by $n_x = n_y = 0$, is non, degenerate:

$$E_0 = \hbar\omega$$

$$\Psi_0 = \psi_0(X)\,\psi_0(Y) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} e^{-\frac{X^2+Y^2}{2}}$$

The second level, $n = n_x + n_y = 1$ is doubly degenerate, $n_x = 1, n_y = 0$; $n_x = 0, n_y =$ ~~$n_x = 0, n_y = 1$~~

and the states are $(\Psi_1)_{10} = \psi_1(X)\,\psi_0(Y) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \sqrt{2}\, X e^{-\frac{X^2+Y^2}{2}}$

with energy $E_1 = 2\hbar\omega$

and $(\Psi_1)_{01} = \psi_0(X)\,\psi_1(Y) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \sqrt{2}\, Y e^{-\frac{X^2+Y^2}{2}}$

and so on.

Now no one product state is an eigenfunction of angular momentum, but any state can be expanded in terms of this set of product states.

First, in order to see what we need, consider the eigenfunctions of the angular momentum operator in polar coordinates of two dimensions. The eigenfunctions are $v(r, \varphi) = R(r)\,\Phi(\varphi)$ since the S-eq. is separable. We seek the form of v which satisfies

$$L_z v = m\hbar v$$

$$\frac{\hbar}{i}\frac{\partial}{\partial\varphi} R\Phi = m\hbar R\Phi$$

$$\frac{d\Phi}{d\varphi} = im\Phi$$

$$\frac{d\Phi}{\Phi} = im\,d\varphi$$

$$\Phi = A e^{im\varphi} \qquad \text{so that } v(r,\varphi) = R(r)\, e^{im\varphi}$$

or, for $m = -1$, $v = R(r)\, e^{-i\varphi}$

We know from the algebra of the complex plane that $X - iY = r e^{-i\varphi}$

where $\begin{cases} r^2 = X^2 + Y^2 \\ \varphi = \tan^{-1}\frac{Y}{X} \end{cases}$

So the eigenstate, expanded in the set of cartesian states, contains only 2 terms:

$$(\Psi_1)_{10} - i(\Psi_1)_{01}$$

that is, the soln is just $v_{\substack{n=1\\m=-1}}(r,\varphi) = \frac{1}{\sqrt{2}}\left[(\Psi_1)_{10} - i(\Psi_1)_{01}\right]$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} (X - iY)\, e^{-\frac{X^2+Y^2}{2}}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} r\, e^{-\frac{r^2}{2}}\, e^{-i\varphi}$$

The energy of this state, as we have seen, is $\underline{E_1 = 2\hbar\omega}$, since the degenerate states occur with equal weight.

$$\mathcal{H} = -H(\mu_1\sigma_{1z} + \mu_2\sigma_{2z}) + c\left[\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}\right]$$

Using σ matrices: (let $\mu = -(\mu_1+\mu_2)$ $\mu' = -(\mu_1-\mu_2)$)

$$\mathcal{H}(\uparrow\uparrow) = -H(\mu_1+\mu_2)(\uparrow\uparrow) + c\left[(\downarrow\downarrow) + (i)(i)(\downarrow\downarrow) + (\uparrow\uparrow)\right] = (H\mu + c)(\uparrow\uparrow)$$

$$\mathcal{H}(\uparrow\downarrow) = -H(\mu_1-\mu_2)(\uparrow\downarrow) + c\left[(\downarrow\uparrow) + (i)(-i)(\downarrow\uparrow) - (\uparrow\downarrow)\right] = (H\mu' - c)(\uparrow\downarrow) + 2c(\downarrow\uparrow)$$

$$\mathcal{H}(\downarrow\uparrow) = -H(-\mu_1+\mu_2)(\downarrow\uparrow) + c\left[(\uparrow\downarrow) + (-i)(i)(\uparrow\downarrow) - (\downarrow\uparrow)\right] = 2c(\uparrow\downarrow) + (-H\mu' - c)(\downarrow\uparrow)$$

$$\mathcal{H}(\downarrow\downarrow) = -H(-\mu_1-\mu_2)(\downarrow\downarrow) + c\left[(\uparrow\uparrow) + (-i)(-i)(\uparrow\uparrow) + (\downarrow\downarrow)\right] = (-H\mu + c)(\downarrow\downarrow)$$

thus
$$H = \begin{pmatrix} H\mu + c & 0 & 0 & 0 \\ 0 & H\mu' - c & 2c & 0 \\ 0 & 2c & -H\mu' - c & 0 \\ 0 & 0 & 0 & -H\mu + c \end{pmatrix}$$

energy levels, ω, are given by:

$$H\psi = \omega\psi \Rightarrow (H - \omega I)\psi = 0 \Rightarrow |H - \omega I| = 0$$

$$\begin{vmatrix} H\mu + c - \omega & & & \\ & H\mu' - c - \omega & 2c & \\ & 2c & -H\mu' - c - \omega & \\ & & & -H\mu + c - \omega \end{vmatrix} = 0$$

$$(H\mu + c - \omega)(-H\mu + c - \omega)\left[(H\mu' - c - \omega)(-H\mu' - c - \omega) - 4c^2\right] = 0$$

$$[\;] = \omega^2 - \omega(H\mu' - c - H\mu' - c) + (H\mu' - c)(-H\mu' - c) - 4c^2$$

$$= \omega^2 + 2\omega c - (H^2\mu'^2 + 3c^2)$$

∴ thus secular equation is:

$$(H\mu + c - \omega)(-H\mu + c - \omega)\left[\omega^2 + 2\omega c - (H^2\mu'^2 + 3c^2)\right] = 0$$

roots are:

$$\omega_1 = c + H\mu$$

$$\omega_3 = c - H\mu$$

$$\omega_{2,4} = -c \pm \sqrt{c^2 + (H^2\mu'^2 + 3c^2)}$$

$$\omega_{2,4} = -c \pm \sqrt{4c^2 + H^2\mu'^2}$$

---

c) for $H \ll \frac{c}{\mu}$; $\sqrt{4c^2 + H^2\mu'^2} \approx 2c$

∴

$$\omega_1 = c + H\mu$$

$$\omega_2 = c$$

$$\omega_3 = c - H\mu$$

$$\omega_4 = -3c$$

for $H \gg \frac{c}{\mu}$; $\sqrt{4c^2 + H^2\mu'^2} \approx H\mu'$

∴

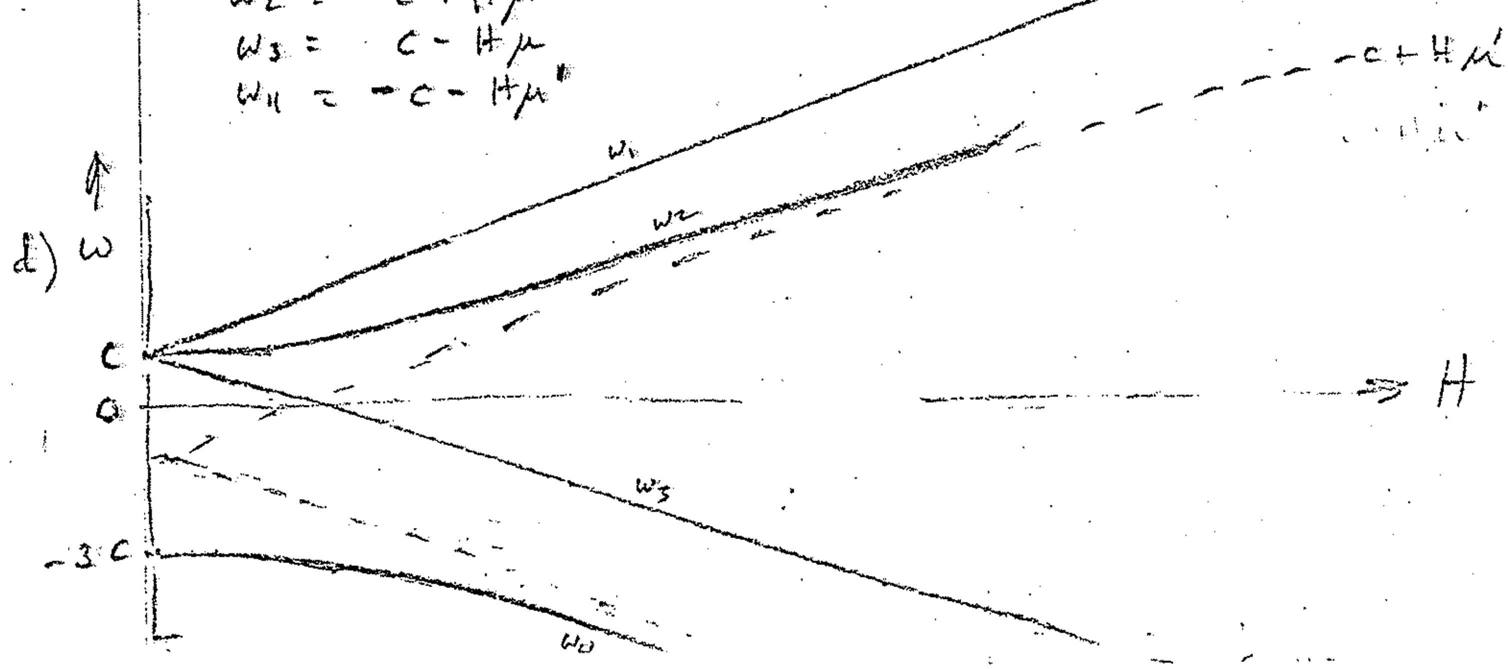
$$\omega_1 = c + H\mu$$

$$\omega_2 = -c + H\mu'$$

$$\omega_3 = c - H\mu$$

$$\omega_4 = -c - H\mu'$$

d)



Consider a beam of electrons, with spin $\frac{1}{2}$. If we are interested in the polarization of the beam only, we have a system with 2 degrees of freedom, so the system is described by a 2×2 density matrix, with 3 independent parameters. The physical structure of these [illegible] completely defined by a knowledge of the vector $\vec{P} = \langle \hat{\sigma} \rangle$ (which has 3 components)

where $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Notice that $Tr(\hat{\sigma}_i) = 0$, $Tr(\hat{\sigma}_i \hat{\sigma}_j)_{i \neq j} = 0$ $\quad Tr(\hat{\sigma}_i^2) = 2$

The most general way to write a 2×2 matrix is thus

$$\hat{\rho} = a\hat{1} + (\vec{a} \cdot \hat{\vec{\sigma}})$$

Now $Tr(\hat{\rho}) = Tr(\hat{\rho}\hat{1}) = \langle \hat{1} \rangle = 1$

So $Tr(\hat{\rho}) = a + a \quad + \underbrace{Tr(a_x \hat{\sigma}_x)}_{\to 0} + \underbrace{Tr(a_y \hat{\sigma}_y)}_{\to 0} + \underbrace{Tr(a_z \hat{\sigma}_z)}_{\to 0}$

$= 2a$

So $1 = 2a$ or $\underline{\underline{a = \frac{1}{2}}}$

also $\vec{P} = \langle \hat{\sigma} \rangle = Tr(\rho \hat{\sigma}) = Tr\,[\underbrace{a\hat{\sigma}}_{\to 0} + (\vec{a} \cdot \hat{\sigma})\hat{\sigma}]$

$= Tr\,[a_x \hat{\sigma}_x (\hat{\sigma}_x \hat{i}) + a_y \hat{\sigma}_y (\hat{\sigma}_y \hat{j}) + a_z \hat{\sigma}_z (\hat{\sigma}_z \hat{k})]$

$= 2\vec{a}$

So $\hat{\rho} = \frac{1}{2}\left[\hat{1} + (\vec{P} \cdot \hat{\vec{\sigma}})\right] = \frac{1}{2}\begin{pmatrix} 1+P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix}$