This Comprehensive Examination for Fall 1972 consists of six problems of equal weight (20 points each). Half the problems are judged to be at intermediate undergraduate level, the other half are graduate level. Work carefully, and show all your steps, so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the blue-book — you may get some credit for it!

If something is omitted from the statement of the problem or you feel there are ambiguities, get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books and papers, except pencil and blue-book on the floor.

Some information you may find useful follows:

\[ \pi = 3.14159 \]

\[(1+\delta)^n = 1+n\delta + \ldots \quad \delta \ll 1\]

\[\rho e^{i\phi} = x+iy\]

\[\int \frac{dx}{\sqrt{1+x^2}} = \tan^{-1}x\]

\[\int \frac{xdx}{\sqrt{1+x^2}} = \sqrt{1+x^2}\]

\[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\]

The first few eigenfunctions for the one-dimensional simple harmonic oscillator of mass \(m\) and frequency \(\omega\) are

\[\psi_0 = \left(\frac{mw}{\hbar} \right)^{1/4} e^{-\frac{x^2}{2}}, \quad \psi_1 = \left(\frac{mw}{\hbar} \right)^{1/4} \sqrt{2} xe^{\frac{-x^2}{2}}, \quad \psi_2 = \left(\frac{mw}{\hbar} \right)^{1/4} \sqrt{2} \left(x^2 - \frac{x^2}{4}\right)e^{\frac{-x^2}{2}}\]

where

\[X = \sqrt{\frac{mw}{\hbar}} x\]
1. a) Explain briefly how a quarter-wave plate is made and how it works. Describe how a quarter-wave plate and a linear polarizer can be used to determine the state of polarization of arbitrarily polarized light.

b) Light traveling in the +x direction passes first through a quarter-wave plate whose fast axis is along the x-axis, and then through a linear polarizer whose transmission axis is +30° from the x-axis. It is found that the intensity of the transmitted light is zero. Describe completely (and justify your statements) the state of polarization of the incident light.

2. A stream of particles of kinetic energy $E$ is moving in the positive x-direction. At $x = 0$ they hit a potential step (see Fig.).

\[ V(x) \]

\[ V = V_1 \]

\[ V = 0 \]

\[ x = 0 \]

\[ x \]

a) Derive the wave functions in the regions $x < 0$ and $x > 0$ for the case when the particle energy is greater than $V_1$.

b) Determine the reflection ($R$) and transmission ($T$) coefficients. ($R$ is the ratio of the reflected current to the incident current, and $T$ is the ratio of the transmitted current to the incident current.)

c) For the case $E = V_1$ what are the values of $R$ and $T$? State your conclusions and briefly sketch the proof without detailed calculations.

3. A particle moves from rest at the origin at time $t = 0$ in an inertial frame of reference under the influence of a constant force $F = mg\hat{i}$ where $\hat{i}$ is the unit vector in the x-direction. In this problem it is not necessary that $v << c$.

a) Calculate the velocity and position of the particle as a function of time.

b) Draw graphs of $v(t)$ and $x(t)$ and compare these curves with the non-relativistic results.

4. A two-dimensional linear simple harmonic oscillator is in an angular momentum eigenstate with eigenvalue $L_z = -h$. Write the normalized eigenfunction, and calculate the eigen-energy of the state. Justify all steps if you want to get credit.
5. Two particles of spin \( \frac{1}{2} \) and magnetic moments \( \mu_1, \mu_2 \) (\( \mu_1 > \mu_2 > 0 \)) interact with each other magnetically, and also with an external magnetic induction field \( \vec{B} \). The Hamiltonian is:

\[
\hat{H} = -(\mu_1 \vec{\sigma}_1 + \mu_2 \vec{\sigma}_2) \cdot \vec{B} + c \vec{\sigma}_1 \cdot \vec{\sigma}_2 \\
\text{c = constant}
\]

a) Take \( \vec{B} \) to be in the \( +z \)-direction. Write the Hamiltonian matrix explicitly in representation that has as basis states: \( (\uparrow\uparrow), (\uparrow\downarrow), (\downarrow\uparrow), (\downarrow\downarrow) \). (Each particle in an eigenstate of \( \sigma_z \).)

b) Derive exact expressions for the energies of the eigenstates of the system.

c) Give approximate expressions for the energy levels valid for (i) small \( B \) (first order in \( B \)) and (ii) very large \( B \).

d) Draw a rough graph showing the energy levels as a function of \( B \).

6. Einstein's Principle of Equivalence says it is impossible to distinguish between the effects of a uniform acceleration and a uniform gravitational field. Two clocks, \( A \) and \( B \), having been synchronized in free space at relative rest, are placed at the top and bottom of a 100 meter high building. Using the Principle of Equivalence, calculate which one runs slower. By how much:

\[
\begin{align*}
&\text{\( R_A \)} \\
&\text{\( + \)} \\
&\text{\( \uparrow \)} \\
&\text{\( \uparrow \)} \\
&\text{\( h \)} \\
&\text{\( \uparrow \)} \\
&\text{\( R \)} \\
&\text{\( B \)} \\
&\text{\( \uparrow \)} \\
&\text{\( \uparrow \)}
\end{align*}
\]

will a time interval measured on the ground (by \( B \)) as one second, differ from that measured by \( A \)?

(In your calculations, assume that any \( v \ll c \) so that only terms of lowest order in \( v/c \) need be kept.)
A uniform gravitational field can exist only over limited regions of space (otherwise bodies could be accelerated indefinitely to acquire infinite momentum). In this case we assume that any velocities that can be acquired are small, i.e.,

Consider two clocks $A$ and $B$ separated by a distance $h$ and subject to a uniform acceleration $g$. Let both clocks sweep by a third clock $C$ in free space (suppose all three clocks were synchronized in free space at rest).

![Diagram of clocks](image)

Suppose the clocks have velocities $v_A$ and $v_B$ as they pass successively the clock $C$ ($v_A$ is $v_B$ since $B$ accelerated through an extra distance $h$). We now apply Special Relativity to find the relation between a time interval $\Delta T_C$ of fixed duration as measured by $C$ and the corresponding intervals $\Delta T_A$ and $\Delta T_B$ as measured by the clocks $A$ and $B$ as they each pass $C$ in time.

Thus $\Delta T_A = \Delta T_C \left(1 - \frac{v_A^2}{c^2}\right)^{-1} \approx \Delta T_C \left(1 + \frac{1}{2} \frac{v_A^2}{c^2}\right)$

and $\Delta T_B = \Delta T_C \left(1 - \frac{v_B^2}{c^2}\right)^{-1} \approx \Delta T_C \left(1 + \frac{1}{2} \frac{v_B^2}{c^2}\right)$

so $\Delta T_B = \Delta T_A \left(1 + \frac{1}{2} \frac{v_A^2}{c^2}\right) \left(1 + \frac{1}{2} \frac{v_B^2}{c^2}\right) \approx \Delta T_A \left(1 - \frac{1}{2} \frac{v_A^2}{c^2}\right) \left(1 + \frac{1}{2} \frac{v_B^2}{c^2}\right) \approx \Delta T_A \left(1 + \frac{1}{2} \frac{v_A^2 + v_B^2}{c^2}\right)$

but since both clocks are accelerating uniformly at $g$, $v_B = v_A + 2gh$

so that $\Delta T_B \approx \Delta T_A \left(1 + \frac{gh}{c^2}\right)$

as that $gh$ has the dimensions of potential energy, so $gh$ has the dimensions of potential.
In fact, the requirement must be
\[
\Delta T = \Delta T_8 \left(1 + \frac{\beta c^2}{c^2} \right) = \Delta T_8 \left(1 + \frac{R_8 - R_8}{c^2} \right) - \frac{R_8}{c^2} \int_{-GM(R_8 - R_8) / c^2}^{R_8} \frac{R_8}{R_8 - R_8} dR_8.
\]

Thus,
\[
\Delta T = \Delta T_8 \left(1 + \frac{\beta c^2}{c^2} \right) = \Delta T_8 \left(1 + \frac{R_8 - R_8}{c^2} \right) - \frac{R_8}{c^2} \int_{-GM(R_8 - R_8) / c^2}^{R_8} \frac{R_8}{R_8 - R_8} dR_8.
\]

The gravitational pull of a uniform acceleration field, the fact that identical objects which keep different times under uniform acceleration (as we have seen) imply that there must exist a difference in acceleration on different planes. It is possible to

\[ \nabla \phi \]

The Blair effect.
a) Explain how a quarter-wave plate and a linear polarizer can be used to determine the state of polarization of arbitrarily polarized light.

b) Light travelling in the +z direction passes through a quarter-wave plate whose fast axis is along the x-axis, and then through a linear polarizer whose transmission axis is +30° from the x-axis. It is found that the intensity of the transmitted light is zero. Describe completely the state of polarization of the incident light.
The residual activity is still unpolarized component.

\[ k = \tan \theta = \frac{2}{3} \]

Elliptical

Polarization

+ x

- x

The fast axis and the polarization are adjusted to minimize transmitted intensity. The angle, \( \theta \), with respect to the x-axis is given by \( \theta = \tan^{-1}\left(\frac{2}{3}\right) \).

Using the ellipse's equation, the x-axis is aligned with the electric field of the x-y plane, the electric field being polarized with the x-y components equal to \( x = -y \times y = x \times y \).

Light out of glass

Axies of the ellipse

+ y

- y

A quarter-wave plate whose fast axis is in the x-y plane is used. The x-axis of the x-y plane is aligned with the x-axis of the x-y plane. The x-axis of the x-y plane is then rotated, with respect to the x-axis of the x-y plane, an angle of \( \pi/4 \).

\( A \)
\[ a, \quad \psi_{x<0} = (A e^{-i k_0 x} + B e^{i k_0 x}) e^{-i \omega t} \]

\[ \omega = \frac{E}{\hbar}, \quad k_0^2 = \frac{2mE}{\hbar^2} \]

\[ \psi_{x>0} = C e^{i k_1 x} e^{-i \omega t} \]

\[ k_1 = \frac{2m(E-V_1)}{\hbar^2} \]

\[ b) \quad \text{and} \quad \frac{d\psi}{dx} \text{ must be continuous at } x = 0 \]

\[ A + B = C, \quad \frac{k_0}{k_1} (A - B) = k_1 C \]

Current densities are \[ j_B = -\frac{p_B}{m} \left( \frac{\hbar k_0}{m} \right) \]

\[ j_C = \frac{p_C}{m} \left( \frac{\hbar k_1}{m} \right) \]

\[ p_B = |B|^2, \quad p_C = |C|^2. \]

\[ \frac{B}{A} = \frac{k_0 - k_1}{k_0 + k_1} \]

\[ \eta = \left| \frac{j_B}{j_A} \right| = \left( \frac{k_0 - k_1}{k_0 + k_1} \right)^2 \]
\[
\frac{v}{A} = \frac{2K_0}{K_1 + K_0} \quad \therefore \quad T = \left| \frac{\frac{\partial}{\partial x}}{\frac{\partial}{\partial x}} \right| = 4 \frac{K_1 K_0^2}{K_0 (K_1 + K_0)^2} = 4 \frac{K_1 K_0}{(K_1 + K_0)}
\]

Note that \( R + T = 1 \)

c) for \( E < V_1 \) \( k_i = c \mu \) \( \mu = \text{real} \).

Thus \( \frac{B}{A} = \frac{1 - \frac{c \mu}{K_0}}{1 + \frac{c \mu}{K_0}} \quad \mu = \sqrt{\frac{8m (V_1 - E)}{\hbar^2}} \)

\[
\left| \frac{\frac{\partial B}{\partial x}}{\frac{\partial A}{\partial x}} \right| = \left| \frac{B}{A} \right|^2 = 1
\]

\( R = 1 \) \( T = 0 \).
2) eq of motion is

\[ \frac{m}{\gamma} \frac{d}{dt} \left[ \frac{\gamma u}{\sqrt{1 - \frac{u^2}{c^2}}} \right] = mg \]

boundary and \( u = 0 \) at \( t = 0 \).

Integration yields

\[ \frac{\gamma u}{\sqrt{1 - \frac{u^2}{c^2}}} = g t \]

\[ u = \frac{g t}{\sqrt{1 + \frac{\gamma^2 v^2}{c^2}}} \]

Note that \( u \leq c \) for all values of \( t \)

and \( u \to c \) as \( t \to \infty \).

\[ \frac{dx}{dl} = \frac{g t}{\sqrt{1 + \frac{\gamma^2 v^2}{c^2}}} \]

\[ X = \int_0^t \frac{g t \, dt'}{\sqrt{1 + \frac{\gamma^2 v^2}{c^2}}} \]

\[ X = \frac{c^2}{g} \left[ \sqrt{1 + \frac{\gamma^2 v^2}{c^2}} - 1 \right] \]

\[ \left( \text{hint) useful deep} \right) \]

\[ \int \frac{\chi \, dx}{\sqrt{1 + x^2}} = \sqrt{1 + x^2} \]
Relativity question

Answer: const

b).

\[ \frac{\gamma}{c} \]

\[ \frac{1}{c} \]

\[ x \]

\[ t \]

non-relativistic

relativistic

non-relativistic

relativistic
A two-dimensional linear harmonic oscillator is in an eigenstate of angular momentum with eigenvalue $\ell = -1$. Write the normalized eigenfunction, and calculate the eigenenergy of the state.

Note: The first few eigenfunctions for the one-dimensional simple harmonic oscillator of mass $m$ and frequency $\omega = \sqrt{\frac{k}{m}}$ are:

$$\psi_0 = \left(\frac{m\omega}{\hbar \pi}\right)^{1/4} e^{-\frac{x^2}{2}}$$
$$\psi_1 = \frac{1}{\sqrt{2}} \psi_0 e^{-\frac{x^2}{2}}$$
$$\psi_2 = \left(\frac{m\omega}{\hbar \pi}\right)^{1/4} \left(\xi x - \frac{x^3}{3}\right) e^{-\frac{x^2}{2}}$$

End note:

Since the one-dimensional eigenfunctions for the SHO, the two-dimensional eigenfunctions are easily constructed as products of the $\psi_n$, since the $S^2$-equation is separable.

Thus, $\psi_0(x,y) = \psi_0(x)\psi_0(y)$, $E_n = (n_x + n_y + 1)\hbar \omega$.

Thus, the lowest energy level, $E_0$, is given by $n_x = n_y = 0$, and hence, degenerate:

$$E_0 = \hbar \omega$$
$$\psi_0 = \psi_0(x)\psi_0(y) = \left(\frac{m\omega}{\hbar \pi}\right)^{1/4} e^{-\frac{x^2 + y^2}{2}}$$

The second level, $n_x = n_y + 1 = 1$, is doubly degenerate, $n_x = 1, n_y = 0$, $n_x = 0, n_y = 1$.

and the states are $\psi_1 = \psi_0(x)\psi_0(y)$ with energy $E_1 = 2\hbar \omega$.

$$\psi_1 = \psi_0(x)\psi_0(y) = \left(\frac{m\omega}{\hbar \pi}\right)^{1/4} \left(\xi x - \frac{x^3}{3}\right) e^{-\frac{x^2}{2}}$$

and so on.
Now we are to consider the eigenfunctions of angular momentum. Any state can be expanded in terms of this set of product states.

First, we order to see what we need; consider the eigenfunctions of the angular momentum operator in polar coordinates of two dimensions. The eigenfunctions are \( \psi(\mathbf{r}, \theta) = R(r) \Phi(\theta) \) since the \( z = \mathbf{r} \) is an expression.

We seek the form of \( \psi \) which satisfies

\[
L_z \psi = m \hbar \psi
\]

\[
\frac{d}{d\theta} R \Phi = m \hbar \Phi
\]

\[
\frac{d}{d\theta} \Phi = i m \hbar \Phi
\]

\[
\Phi = A e^{i m \phi}
\]

So that \( \psi(\mathbf{r}, \theta) = R(r) e^{i m \phi} \)

or, for \( m = -1 \), \( \psi = R(r) e^{-i \phi} \)

We know from the algebra of the complex plane that \( \chi - i \xi = \pi e^{-i \phi} \)

\[
\frac{\pi}{2} = \frac{\xi^2 + \chi^2}{\chi i}
\]

\( \phi = \tan^{-1} \frac{\chi}{\xi} \)

So the eigenstates expanded as the set of cartesian states contains only two terms:

\[
(\Phi_0)^{-1} - i (\Phi_1)^{-1}
\]

That is, the solution is given

\[
\psi(\mathbf{r}, \theta) = \frac{1}{2} [ (\Phi_0)^{-1} - i (\Phi_1)^{-1} ]
\]

\[
= \left( \frac{\mu \omega}{\hbar} \right)^{\frac{1}{4}} \left( \chi - i \xi \right) e^{-\frac{\pi^2 + \chi^2}{4}}
\]

\[
= \left( \frac{\mu \omega}{\hbar} \right)^{\frac{1}{4}} \chi e^{\frac{\pi^2}{4}} e^{-i \phi}
\]

The energy of this state, as we have seen, is \( E = 2 \hbar \omega \), since the degenerate states occur with equal weight.
\[ H = - \mathcal{H}(\mu_i, \sigma_{1z} + \mu_z \sigma_{2z}) + C \left[ \sigma_{1x} \sigma_{2x} + \sigma_{1y} \sigma_{2y} + \sigma_{1z} \sigma_{2z} \right] \]

Using the relations: (let \( \mu = -(\mu_i + M \lambda) \) \( \mu' = -3(\mu_i + M \lambda) \))

\[ H(\mu) = -\mathcal{H}(\mu, \mu) + C \left[ (M^2 + (\sigma_i \sigma_i) (\mu) + (M^2) \right] = (\mu \mu' + C) (M^2) \]

\[ H(\mu') = -\mathcal{H}(\mu, -\mu) + C \left[ (\mu')^2 + (\sigma_i \sigma_i) (\mu') = (\mu \mu' - C) (M^2) + 2C (M^2) \right] \]

\[ H(\mu) = -\mathcal{H}(\mu, -\mu) + C \left[ (\sigma_i \sigma_i) (\mu) = 2C (M^2) + (-\mu \mu' - C) (M^2) \right] \]

\[ H(\mu') = -\mathcal{H}(\mu, -\mu) + C \left[ (\sigma_i \sigma_i) (\mu') = C - \mu \mu' + C) (M^2) \right] \]

Thus:

\[ H = \begin{pmatrix} H_{\mu + C} & 0 & 0 & 0 \\ 0 & H_{\mu - C} & 2C & 0 \\ 0 & 2C & -H_{\mu - C} & 0 \\ 0 & 0 & 0 & -H_{\mu + C} \end{pmatrix} \]

Among these, \( \omega, \omega \), we obtain:

\[ H \phi = \omega \phi \Rightarrow (H - \omega I) \phi = 0 \Rightarrow \left| H - \omega I \right| = 0 \]

\[ \begin{vmatrix} H_{\mu + C} - \omega & \omega & 0 & 0 \\ 0 & H_{\mu - C} - \omega & 2C & 0 \\ 0 & 2C & -H_{\mu - C} - \omega & 0 \\ 0 & 0 & 0 & -H_{\mu + C} - \omega \end{vmatrix} = 0 \]

\[ \begin{vmatrix} (H_{\mu + C} - \omega)(-H_{\mu + C} - \omega) \left[ (H_{\mu - C} - \omega)(-H_{\mu - C} - \omega) - 4C^2 \right] \end{vmatrix} = 0 \]

\[ \begin{vmatrix} \omega^2 - \omega \left( H_{\mu - C} - H_{\mu - C} \right) + (H_{\mu - C}(-H_{\mu - C}) - 4C^2 \right) \end{vmatrix} = 0 \]

\[ \omega^2 + 2 \omega C = -\left( \frac{\omega^2}{2} + 2C \right) \]
Thus secular equation is:

$$(
\mu \pm c - \omega)(
-\mu \pm c - \omega)
\left[
\omega^2 + 2\nu c - (\mu^2 + 3c^2)
\right] = 0$$

Roots are:

$\omega_1 = c + \mu$

$\omega_2 = c - \mu$

$$\omega_3 = -c \pm \sqrt{c^2 + (\mu^2 + 3c^2)}$$

$$\omega_4 = -c \pm \sqrt{\frac{1}{4}c^2 + \mu^2}$$

For $H > \frac{\mu}{c}$, $\sqrt{c^2 + \mu^2} \approx 2c$

$\omega_1 = c + \mu$

$\omega_2 = c$

$\omega_3 = c - \mu$

$\omega_4 = -3c$

$\omega_1 = c + \mu$

$\omega_2 = c + \mu$

$\omega_3 = c - \mu$

$\omega_4 = -c - \mu$
Consider a beam of electrons, with spin $\frac{1}{2}$. If we are interested in the polarization of the beam only, we have a system with 2 degrees of freedom, so the system is described by a $2 \times 2$ density matrix, with 3 independent parameters. The physical activation of these parameters is a vector completely defined by a knowledge of $\rho = \langle \hat{S} \rangle$ (which has 3 components)

where \( \hat{S}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( \hat{S}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \), \( \hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

Note that \( T_n(\hat{S}_x) = 0 \), \( T_n(\hat{S}_y) = 0 \), \( T_n(\hat{S}_z) = 2 \).

The most general way to write a $2 \times 2$ matrix is then

\[ \hat{\rho} = a \hat{1} + (\hat{S}_x \cdot \hat{S}) \]

Note that \( T_n(\hat{1}) = T_n(\hat{S}_x) = T_n(\hat{S}_y) = T_n(\hat{S}_z) = 1 \).

Thus \( T_n(\hat{S}) = a + \hat{a} + T_n(a \hat{S}_x) + T_n(a \hat{S}_y) + T_n(a \hat{S}_z) \to 0 \)

\[ = 2a \]

So \( 1 + 2a = 0 \) or \( a = \frac{1}{2} \).

Also \( \rho = \langle \hat{S} \rangle = T_n(\hat{S}) = T_n(\hat{S}) + T_n(\hat{S}_x) + T_n(\hat{S}_y) + T_n(\hat{S}_z) \]

\[ = T_n \left[ a \hat{S}_x (\hat{S}_x \cdot \hat{S}) + a \hat{S}_y (\hat{S}_y \cdot \hat{S}) + a \hat{S}_z (\hat{S}_z \cdot \hat{S}) \right] \]

\[ = 2 \hat{a} \]

So \( \hat{S} = \frac{1}{2} \left[ \hat{I} + (\rho, \hat{S}) \right] = \frac{1}{2} \left( \begin{pmatrix} 1 & P \omega \\ P \omega & 1-P \end{pmatrix} \right) \).