OSU Physics Department
Comprehensive Examination #130

Monday, January 8 and Tuesday, January 9, 2018

Winter 2018 Comprehensive Examination

PART 1, Monday, January 8, 9:00am
PART 2, Monday, January 8, 1:00pm
PART 3, Tuesday, January 9, 9:00am
PART 4, Tuesday, January 9, 1:00pm

General Instructions

This Winter 2018 Comprehensive Examination consists of four separate parts of two problems each. Each problem caries equal weight (20 points each) and lasts three hours. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit–especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all exams and formula sheets at the end of the exam.

A harmonic oscillator of mass m and angular frequency ω is in a superposition of the first and second excited states at $t = 0$,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right),$$

where the states $|1\rangle$ and $|2\rangle$ have the energy eigenvalues of $\frac{3}{2}\hbar\omega$ and $\frac{5}{2}\hbar\omega$, respectively.

(a) Find the state vector $|\psi(t)\rangle$ at time t. What is the probability that the harmonic oscillator is in the state $|2\rangle$ at time t?

(b) Calculate the position and momentum expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ at time t.

(c) What is the energy expectation value $\langle\hat{H}\rangle$ at time t? Does your answer have any time dependence? Make a brief explanation of the physical meaning for your answer which is either 'yes' or 'no'.

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where the states $|1\rangle$ and $|2\rangle$ have the energy eigenvalues of $\frac{3}{2}\hbar\omega$ and $\frac{5}{2}\hbar\omega$, respectively.

(a) Find the state vector $|\psi(t)\rangle$ at time t. What is the probability that the harmonic oscillator is in the state $|2\rangle$ at time t?

Solution:
The states $|1\rangle$ and $|2\rangle$ are the eigenstates of the Hamiltonian,

$$\hat{H}|1\rangle = E_1|1\rangle, \qquad E_1 = \frac{3}{2}\hbar\omega \tag{1}$$

$$\hat{H}|2\rangle = E_2|2\rangle, \qquad E_1 = \frac{5}{2}\hbar\omega \tag{2}$$

The time evolution of the state vector is

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\frac{\hat{H}}{\hbar}t}|\psi(0)\rangle \\ &= \frac{1}{\sqrt{2}}\left(e^{-i\frac{E_1}{\hbar}t}|1\rangle + e^{-i\frac{E_2}{\hbar}t}|2\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(e^{-i\frac{3}{2}\omega t}|1\rangle + e^{-i\frac{5}{2}\omega t}|2\rangle\right) \end{aligned} \tag{3}$$

The probability for the harmonic oscillator in $|2\rangle$ at time t is

$$P_2 = \left|\frac{1}{\sqrt{2}}e^{-i\frac{5}{2}\omega t}\right|^2 = \frac{1}{2} \tag{4}$$

(b) Calculate the position and momentum expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ at time t.

Solution:
The position operator can be expressed as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \tag{5}$$

Then, the position expectation value at time t is

$$
\begin{aligned}
\langle \hat{x} \rangle(t) &= \langle \psi(t)|\hat{x}|\psi(t)\rangle \\
&= \frac{1}{2}\left(\langle 1| + e^{i\omega t}\langle 2|\right)\left\{\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)\right\}\left(|1\rangle + e^{-i\omega t}|2\rangle\right) \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 1| + e^{i\omega t}\langle 2|\right)\Big(|0\rangle + \sqrt{2}|2\rangle + \sqrt{2}e^{-i\omega t}|1\rangle + \sqrt{3}e^{-i\omega t}|3\rangle\Big) \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{2}e^{-i\omega t} + \sqrt{2}e^{i\omega t}) \\
&= \sqrt{\frac{\hbar}{m\omega}}\cos\omega t \qquad (6)
\end{aligned}
$$

Similarly, the momentum expectation value at time t is

$$
\begin{aligned}
\langle \hat{p} \rangle(t) &= \langle \psi(t)|\hat{p}|\psi(t)\rangle, \\
&= \frac{1}{2}\left(\langle 1| + e^{i\omega t}\langle 2|\right)\left\{\frac{1}{i}\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}-\hat{a}^\dagger)\right\}\left(|1\rangle + e^{-i\omega t}|2\rangle\right) \\
&= \frac{1}{2i}\sqrt{\frac{m\hbar\omega}{2}}\left(\langle 1| + e^{i\omega t}\langle 2|\right)\Big(|0\rangle - \sqrt{2}|2\rangle + \sqrt{2}e^{-i\omega t}|1\rangle - \sqrt{3}e^{-i\omega t}|3\rangle\Big) \\
&= \frac{1}{2i}\sqrt{\frac{m\hbar\omega}{2}}(\sqrt{2}e^{-i\omega t} - \sqrt{2}e^{i\omega t}) \\
&= -\sqrt{m\hbar\omega}\sin\omega t \qquad (7)
\end{aligned}
$$

(c) What is the energy expectation value $\langle \hat{H} \rangle$ at time t? Does your answer have any time dependence? Make a brief explanation of the physical meaning for your answer which is either 'yes' or 'no'.

Solution:
The energy expectation value is

$$
\begin{aligned}
\langle \hat{H} \rangle &= \langle \psi(t)|\hat{H}|\psi(t)\rangle, \\
&= \frac{1}{2}\left(\langle 1| + e^{i\omega t}\langle 2|\right)\left(E_1|1\rangle + E_2 e^{-i\omega t}|2\rangle\right) \\
&= \frac{1}{2}(E_1 + E_2) = 2\hbar\omega, \qquad (8)
\end{aligned}
$$

which is independent of time. The energy must be conserved in the system of time-independent Hamiltonian.

The Hamiltonian for a spin one system in an electric field E and a magnetic field B is given as

$$\hat{H} = \hat{H}_0 + \hat{H}_1 = \begin{pmatrix} -\mu B & 0 & QE \\ 0 & 0 & 0 \\ QE & 0 & \mu B \end{pmatrix}, \quad \mu B > 0$$

where μ is the magnetic dipole moment of the spin one system and Q is a constant including the electric quadrupole moment.

(a) Determine the eigenvalues by means of perturbation theory, when $QE \ll \mu B$. Treat the electric field E as a small parameter and calculate to second order in E.

(b) Calculate the exact eigenvalues by diagonalizing the matrix and show that they are consistent with the results of (a).

(c) What happens when $B = 0$? Determine the eigenstates and eigenvalues.

The Hamiltonian for a spin one system in an electric field E and a magnetic field B is given as

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where μ is the magnetic dipole moment of the spin one system and Q is a constant including the electric quadrupole moment.

(a) Determine the eigenvalues by means of perturbation theory, when $QE \ll \mu B$. Treat the electric field E as a small parameter and calculate to second order in E.

Solution:
When $QE \ll \mu B$, the Hamiltonian can be expressed as $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$H_0 = \begin{pmatrix} -\mu B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu B \end{pmatrix} \text{ and } \hat{H}_1 = \begin{pmatrix} 0 & 0 & QE \\ 0 & 0 & 0 \\ QE & 0 & 0 \end{pmatrix} \tag{9}$$

with the zeroth order energy eigenvalues $E_k^{(0)} = -\mu B, 0, \mu B$ and the basis set,

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{10}$$

The first order energy correction is

$$E_k^{(1)} = \langle k|\hat{H}_1|k\rangle = 0 \text{ for } k = 1, 0, -1 \tag{11}$$

The second order energy correction is

$$E_k^{(2)} = \sum_{n \neq k} \frac{|\langle n|\hat{H}_1|k\rangle|^2}{E_k - E_n}, \tag{12}$$

where $E_1^{(0)} = -\mu B$, $E_0^{(0)} = 0$, and $E_{-1}^{(0)} = \mu B$, and hence

$$E_1^{(2)} = \frac{|\langle 1|\hat{H}_1|-1\rangle|^2}{E_1 - E_{-1}} = -\frac{Q^2}{2\mu B}E^2 \tag{13}$$

$$E_0^{(2)} = \sum_{n \neq 0} \frac{|\langle n|\hat{H}_1|0\rangle|^2}{E_0 - E_n} = 0 \tag{14}$$

$$E_{-1}^{(2)} = \frac{|\langle -1|\hat{H}_1|1\rangle|^2}{E_{-1} - E_1} = \frac{Q^2}{2\mu B}E^2 \tag{15}$$

(b) Calculate the exact eigenvalues by diagonalizing the matrix and show that they are consistent with the results of (a).

Solution:
Diagonalizing the Hamiltonian, we obtain

$$
\begin{aligned}
& |\hat{H} - \lambda \hat{I}| = 0 \\
\Rightarrow \quad & \begin{vmatrix} -\mu B - \lambda & 0 & QE \\ 0 & -\lambda & 0 \\ QE & 0 & \mu B - \lambda \end{vmatrix} = 0 \\
\Rightarrow \quad & -\lambda(-\mu B - \lambda)(\mu B - \lambda) + \lambda Q^2 E^2 = 0 \\
\Rightarrow \quad & \lambda \left[\lambda^2 - Q^2E^2 - \mu^2 B^2\right] = 0 \\
\Rightarrow \quad & \lambda = 0, \lambda = \pm\sqrt{Q^2E^2 + \mu^2 B^2}
\end{aligned}
$$

In the weak E-field limit, $QE \ll \mu B$, a Taylor series approximation leads to

$$\lambda = 0, \lambda = \pm \mu B \sqrt{1 + \frac{Q^2E^2}{\mu^2 B^2}} \cong \pm \left(\mu B + \frac{Q^2}{2\mu B} E^2\right), \tag{16}$$

i.e., the energy eigenvalues up to the second order of E are

$$E_1 \cong -\mu B - \frac{Q^2}{2\mu B} E^2 \tag{17}$$

$$E_0 = 0 \tag{18}$$

$$E_{-1} \cong \mu B + \frac{Q^2}{2\mu B} E^2 \tag{19}$$

This is consistent with the result of (a).

(c) What happens when $B = 0$? Determine the eigenstates and eigenvalues.

Solution:
When $B = 0$, the Hamiltonian is

$$\hat{H} = \begin{pmatrix} 0 & 0 & QE \\ 0 & 0 & 0 \\ QE & 0 & 0 \end{pmatrix} \tag{20}$$

Diagonalizing the Hamiltonian, we get

$$
\begin{aligned}
& |\hat{H} - \lambda \hat{I}| = 0 \\
\Rightarrow \quad & \begin{vmatrix} -\lambda & 0 & QE \\ 0 & -\lambda & 0 \\ QE & 0 & -\lambda \end{vmatrix} = 0 \\
\Rightarrow \quad & \lambda(\lambda^2 - Q^2E^2) = 0 \\
\Rightarrow \quad & \lambda = 0, \lambda = \pm QE
\end{aligned}
$$

Therefore, the energy eigenvalues are $E_+ = QE$, $E_0 = 0$, $E_- = -QE$. The corresponding eigenstates are

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{21}$$

The spin states are degenerated when no magnetic field is applied ($B = 0$) and hence we should apply the degenerate perturbation to find energy corrections. The degenerate perturbation theory is practically identical with the process to get the exact solution by diagonalizing the Hamiltonian in this 3 dimensional space.

Background: A Carnot engine consists of a substance (usually a gas) operating reversibly between two temperatures $T_2 > T_1$. A Carnot cycle consists of four steps: isothermal expansion at T_2 where it absorbs heat of Q_2, adiabatic cooling down to T_1, isothermal compression where it rejects heat of Q_1, and adiabatic heating up to T_2. Note that a cycle means all the thermal dynamic properties, such as pressure, temperature, and volume return to original values.

Carnot engine has the highest efficiency $\eta = \frac{W}{Q_2} = \frac{T_2 - T_1}{T_2}$ for all engines operating between T_1 and T_2, where W is the work done by the engine. In this problem, we are going to test that a slight deviation from Carnot cycle indeed lowers the efficiency. We are going to assume using monatomic ideal gas as our substance.

Monatomic ideal gas follow the ideal gas law $PV = Nk_BT$, where P is the pressure, V is the volume, N is the number of molecules, T is the temperature, and k_B is the Boltzmann constant. Each gas molecule is a single atom without internal structure. The internal energy U of a monatomic ideal gas is the summation of the kinetic energies from all molecules, $U = \frac{3}{2}Nk_BT$.

Problem: Our engine, call it engine X, also consists of monatomic ideal gas. It goes through the following steps that make one cycle: (1) starting at pressure P_2 and volume V_2, the ideal gas goes through isothermal expansion at T_2 until its volume becomes V_m, (2) adiabatic cooling down to T_m, (3) isothermal expansion at T_m until its volume is V_1, (4) adiabatic cooling down to T_1, (5) isothermal compression at T_1, (6) adiabatic compress to T_2 so that the gas returns to the $[P_2, V_2, T_2]$ state. All steps are quasi-static (reversible).

(a) For monatomic ideal gas the adiabatic process is characterized by $P \cdot V^{\gamma_1} =$ constant and $T \cdot V^{\gamma_2} =$ constant. Find γ_1 and γ_2.

(b) Calculate the change of entropy in each step and draw the cycle of engine X on T-S (temperature-entropy) diagram.

(c) Calculate the total work done by engine X in one cycle.

(d) Calculate the efficiency η_X of engine X. Prove that $\eta_X < \frac{T_2 - T_1}{T_2}$. Efficiency of a thermal engine is defined as the total work done (steps 1-6 for engine X) divided by total heat absorbed (step 1 and 3 for engine X).

Background: A Carnot engine consists of a substance (usually a gas) operating reversibly between two temperatures $T_2 > T_1$. A Carnot cycle consists of four steps: isothermal expansion at T_2 where it absorbs heat of Q_2, adiabatic cooling down to T_1, isothermal compression where it rejects heat of Q_1, and adiabatic heating up to T_2. Note that a cycle means all the thermal dynamic properties, such as pressure, temperature, and volume return to original values.

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(a) For monatomic ideal gas the adiabatic process is characterized by $P \cdot V^{\gamma_1} =$ constant and $T \cdot V^{\gamma_2} =$ constant. Find γ_1 and γ_2.

Solution:
The internal energy from equal partition theorem is $U = \frac{3}{2}Nk_BT$. The state equation for isothermal process is simply $PV = Nk_BT$. The state equation for adiabatic process can be obtained from first law: $dU + PdV = 0$, which gives $\frac{3}{2}Nk_BdT + PdV = 0$. Plugging into the ideal gas law we find

$$\frac{3}{2}d(PV) + PdV = 0 \tag{22}$$

$$\frac{3}{2}VdP + \frac{5}{2}PdV = 0 \tag{23}$$

$$PV^{\frac{5}{3}} = constant, TV^{\frac{2}{3}} = constant \tag{24}$$

Namely $\gamma_1 = \frac{5}{3}$, $\gamma_2 = \frac{2}{3}$.

(b) Calculate the change of entropy in each step and draw the cycle of engine X on T-S (temperature-entropy) diagram.

Solution:
The entropy does not change during adiabatic process.

To calculate the change of entropy for isothermal process, we note that

$$dU = dQ - dW = TdS - PdV = 0 \tag{25}$$

Therefore

$$\Delta S = \frac{W}{T} \tag{26}$$

The work done for isothermal process is

$$W = \int_{V_1}^{V_2} PdV = Nk_B T ln \frac{V_2}{V_1} \tag{27}$$

So the change of entropy in each step is:

$$\Delta S_1 = Nk_B ln \frac{V_m}{V_2},\ \Delta S_2 = 0,\ \Delta S_3 = Nk_B ln[\frac{V_1}{V_m}(\frac{T_m}{T_2})^{\frac{3}{2}}] \tag{28}$$

$$\Delta S_4 = 0,\ \Delta S_5 = -Nk_B ln[\frac{V_1}{V_2}(\frac{T_m}{T_2})^{\frac{3}{2}}],\ \Delta S_6 = 0 \tag{29}$$

Note that $\Delta S_5 = -(\Delta S_1 + \Delta S_3)$, such that the cycle closes correctly in the T-S diagram.

(c) Calculate the total work done by engine X in one cycle.

Solution:
The total work done, based on the first law, is

$$W = Q_1 + Q_3 + Q_5 \tag{30}$$

$$= T_2\Delta S_1 + T_m \Delta S_3 + T_1 \Delta S_5 \tag{31}$$

Where the change of entropy follows (b).

(d) Calculate the efficiency η_X of engine X. Prove that $\eta_X < \frac{T_2 - T_1}{T_2}$. Efficiency of a thermal engine is defined as the total work done (steps 1-6 for engine X) divided by total heat absorbed (step 1 and 3 for engine X).

Solution:
It is simple to show that

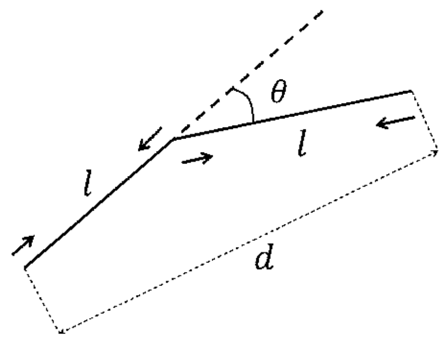
$$\eta_X = \frac{T_2\Delta S_1 + T_m\Delta S_3 - T_1(\Delta S_1 + \Delta S_3)}{T_2\Delta S_1 + T_m \Delta S_3} \tag{32}$$

$$< \frac{T_2\Delta S_1 + T_m\Delta S_3 - T_1\Delta S_1}{T_2\Delta S_1 + T_m\Delta S_3} \tag{33}$$

$$< \frac{T_2\Delta S_1 - T_1\Delta S_1}{T_2\Delta S_1} = \frac{T_2 - T_1}{T_2} \tag{34}$$

A polymer naturally coils up in water even though they have a finite rigidity to resist bending. In this problem, we will try to understand this phenomenon by analyzing a simple model in statistical mechanics.

Consider two rigid rods of equal length l. They are connected at the ends by a joint. The joint has a torsional spring such that if the two rods make an angle θ, the spring stores an energy of $\frac{1}{2}k\theta^2$. Assuming this two-segment polymer is in equilibrium with a heat bath of temperature T. We will also ignore the kinetic energy of the rods.



(a) Calculate the partition function, assuming that $e^{-\frac{k\pi^2}{2k_BT}}$ is negligibly small. Note that the formula of Gaussian integral $\int_{-\infty}^{\infty} e^{-a(x+b)^2}dx = \sqrt{\frac{\pi}{a}}$ will be useful.

(b) Calculate the entropy and internal energy at equilibrium.

(c) Calculate $\langle d^2 \rangle$, where d is the end-to-end distance of the model polymer, and $\langle\rangle$ means ensemble average. Assuming the angle θ is small. Show that indeed a finite temperature causes the polymer to coil up.

(d) Calculate the average end-to-end distance $\sqrt{\langle d^2 \rangle}$ of the model polymer at very high temperature limit.

A polymer naturally coils up in water even though they have a finite rigidity to resist bending. In this problem, we will try to understand this phenomenon by analyzing a simple model in statistical mechanics.

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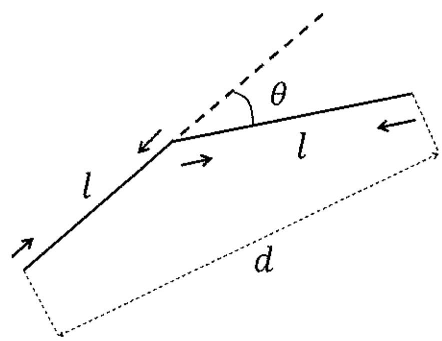


(a) Calculate the partition function, assuming that $e^{-\frac{k\pi^2}{2k_BT}}$ is negligibly small. Note that the formula of Gaussian integral $\int_{-\infty}^{\infty} e^{-a(x+b)^2}dx = \sqrt{\frac{\pi}{a}}$ will be useful.

Solution:
By definition, the partition function equals to

$$\begin{aligned} Z &= \int_{-\pi}^{\pi} e^{-\frac{k\theta^2}{2k_BT}} d\theta && (35) \\ &\approx \int_{-\infty}^{\infty} e^{-\frac{k\theta^2}{2k_BT}} d\theta && (36) \\ &= \sqrt{\frac{2\pi k_BT}{k}} && (37) \end{aligned}$$

(b) Calculate the entropy and internal energy at equilibrium.

Solution:
The energy and entropy can be obtained from the partition function: $Z = e^{-\frac{A}{k_BT}}$, where A is the Helmholtz free energy.

$$\begin{aligned} S &= -\frac{\partial A}{\partial T} = \frac{\partial (k_BTlnZ)}{\partial T} && (38) \\ &= \frac{k_B}{2}(1 + ln\frac{2\pi k_BT}{k}) && (39) \\ & && (40) \end{aligned}$$

To find the internal energy, there are at least three different ways:

$$
\begin{aligned}
U &= A + TS, \text{ or} && (41)\\
&= -\partial_\beta lnZ, \text{ or} && (42)\\
&= \langle \frac{k\theta^2}{2} \rangle_{\text{Boltzmann distribution}} && (43)
\end{aligned}
$$

All the above give the same result, $U = \frac{1}{2}k_BT$. It is not surprising that the internal energy just equals to $\frac{1}{2}k_BT$, because of the equal partition theorem.

(c) Calculate $\langle d^2 \rangle$, where d is the end-to-end distance of the model polymer, and $\langle\rangle$ means ensemble average. Assuming the angle θ is small. Show that indeed a finite temperature causes the polymer to coil up.

Solution:
The average distance squared equals to

$$
\begin{aligned}
\langle d^2 \rangle &= \langle 2l^2 + 2l^2 \cos\theta \rangle && (44)\\
&= \langle 4l^2 - l^2\theta^2 \rangle && (45)\\
&= 4l^2 - \frac{l^2 k_B T}{k} && (46)
\end{aligned}
$$

Clearly $\langle d^2 \rangle < (2l)^2$.

(d) Calculate the average end-to-end distance $\sqrt{\langle d^2 \rangle}$ of the model polymer at very high temperature limit.

Solution:
At high temperature θ is uniformly distributed, therefore we have

$$
\begin{aligned}
\langle d^2 \rangle &= \langle 2l^2 + 2l^2 \cos\theta \rangle && (47)\\
&= 2l^2 && (48)
\end{aligned}
$$

Therefore $\sqrt{\langle d^2 \rangle} = \sqrt{2}l$

A mass m is released from rest and falls through a viscous medium under the combined influence of gravity and a viscous retarding force. You want to test a model that posits that the retarding force is proportional to velocity and directed opposite to the direction of motion: $F_\gamma = -\gamma v$, where γ is a positive constant. You might compare the position and velocity of the mass calculated from this model to data obtained from a video, for example.

(a) Find an expression for the velocity of the mass as a function of time, and show a graphical sketch.

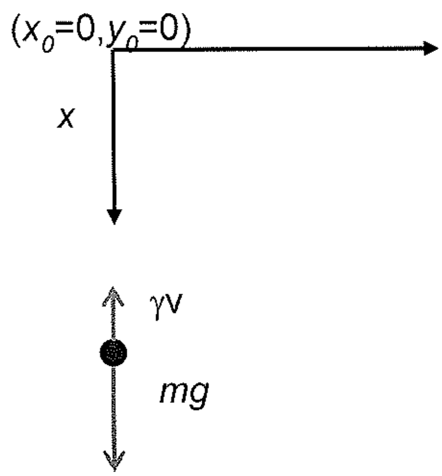
(b) Find an expression for the position of the mass as a function of time and show a graphical sketch.

(c) Demonstrate that the expressions have sensible limits and are consistent with any assumptions. Discuss features of the motion in viscous medium compared to the motion without it.

Comprehensive Exam, Winter 2018, Undergraduate CM Solution

A mass m is released from rest and falls through a viscous medium under the combined influence of gravity and a viscous retarding force. You want to test a model that posits that the retarding force is proportional to velocity and directed opposite to the direction of motion: $F_r = -\gamma v$, where γ is a positive constant. You might compare the position and velocity of the mass calculated from this model to data obtained from a video, for example.

(a) Find an expression for the velocity of the mass as a function of time, and show a graphical sketch. Check that the expression has sensible limits, and point out
(b) Find an expression for the position of the mass as a function of time and show a graphical sketch.
(c) Demonstrate that the expressions have sensible limits and/or are consistent with any assumptions. Discuss features of the motion in viscous medium compared to the motion without it.



This is a 1-dimensional problem. The particle falls from rest, so a convenient origin of the coordinate system is the initial position. Let gravity define the positive direction, and call it x. Then $x_0 = 0$ and increasing x means the particle falls towards the earth. Also $v_0 = 0$. The particle mass is m, the gravitational constant $g > 0$, and the coefficient $\gamma > 0$. The force is then

$$F = mg - \gamma v$$

(a)

Newton's law: $\dfrac{dp}{dt} = F$

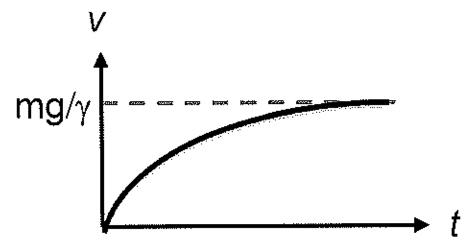
$$m\frac{dv}{dt} = mg - \gamma v \Rightarrow \frac{dv}{g - \frac{\gamma}{m}v} = -dt$$

Integrate (primes are dummy variables):

$$\int_{v_0=0}^{v}\frac{dv'}{g-\frac{\gamma}{m}v'}=\int_0^t dt' \Rightarrow -\frac{m}{\gamma}\ln\left(g-\frac{\gamma}{m}v'\right)\Big|_0^v = t'|_0^t \Rightarrow \ln\left(g-\frac{\gamma}{m}v\right)-\ln(g)=-\frac{\gamma}{m}t \Rightarrow$$

$$\ln\left(\frac{g-\frac{\gamma}{m}v}{g}\right)=-\frac{\gamma}{m}t \Rightarrow g-\frac{\gamma}{m}v=ge^{-\frac{\gamma}{m}t} \Rightarrow$$

$$\boxed{v(t)=\frac{mg}{\gamma}\left(1-e^{-\frac{\gamma}{m}t}\right)}$$



Check limits:

$t=0$: $v(0)=\frac{mg}{\gamma}(1-1)=0$ as assumed.

$t\to 0$: $v=\frac{mg}{\gamma}\left(1-1+\frac{\gamma}{m}t-\frac{1}{2}\left(\frac{\gamma}{m}t\right)^2+\ldots\right)$ so for times short compared to m/γ, the velocity increases linearly, which looks like the motion under constant gravitational force. At longer times, the velocity becomes large enough to make the retarding force significant, and the value becomes smaller than it would be without the viscous fluid.

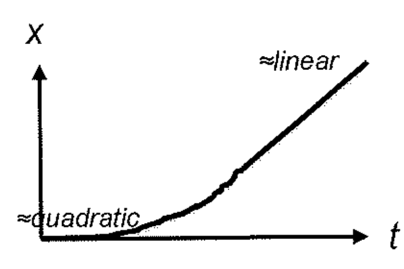
$t\to\infty$: $v(\infty)=\frac{mg}{\gamma}(1-0)=\frac{mg}{\gamma}$ which means that the velocity becomes constant after a time long compared with m/γ. We call this "terminal velocity" $v_{term}=\frac{mg}{\gamma}$. This constant velocity is in contrast to the case of no viscosity where the velocity increases linearly, forever. Terminal velocity can also be found by setting the net force equal to zero (where the gravitational force is balanced by the viscous force).

(b) Integrate velocity equation:

$$v(t)=\frac{dx}{dt}=\frac{mg}{\gamma}\left(1-e^{-\frac{\gamma}{m}t}\right) \Rightarrow dx=\frac{mg}{\gamma}\left(1-e^{-\frac{\gamma}{m}t}\right)dt \Rightarrow$$

$$\int_{x(t=0)}^{x}dx'=\frac{mg}{\gamma}\int_0^t\left(1-e^{-\frac{\gamma}{m}t}\right)dt' \Rightarrow x=\frac{mg}{\gamma}\int_0^t dt'-\frac{mg}{\gamma}\int_0^t e^{-\frac{\gamma}{m}t'}dt' \Rightarrow$$

$$\boxed{x(t)=\frac{mg}{\gamma}t+\frac{m^2g}{\gamma^2}\left(e^{-\frac{\gamma}{m}t}-1\right)}$$



Check limits:

$t=0$: $x(0)=\frac{mg}{\gamma}0+\frac{m^2g}{\gamma^2}(1-1)=0$ as assumed.

$t\to\infty$: $x\xrightarrow{t\to\infty}\frac{mg}{\gamma}\underset{\to\text{large}}{t}+\underbrace{\frac{m^2g}{\gamma^2}(0-1)}_{\text{small cf 1st term}}\xrightarrow{t\to\infty}v_{term}t$ which means that the position increases linearly after a long time, which is what you expect from a constant velocity.

If the viscous force tends to zero, you expect the standard result to recovered if (expand the exponential).

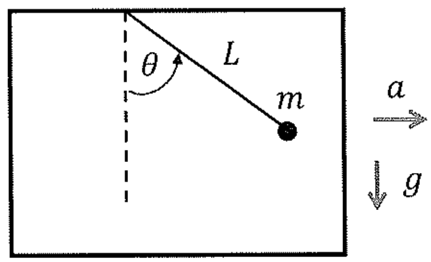
$$x=\frac{mg}{\gamma}t+\frac{m^2g}{\gamma^2}\left(e^{-\frac{\gamma}{m}t}-1\right)=\frac{mg}{\gamma}t+\frac{m^2g}{\gamma^2}\left(\cancel{1}-\frac{\gamma}{m}t+\frac{1}{2}\left(\frac{\gamma}{m}t\right)^2+\ldots\cancel{-1}\right)$$

$$x=\frac{mg}{\gamma}t+\frac{m^2g}{\gamma^2}\left(-\frac{\gamma}{m}t+\frac{1}{2}\left(\frac{\gamma}{m}t\right)^2\right)=\underbrace{\frac{mg}{\gamma}t-\frac{mg}{\gamma}t}_{cancel}+\frac{gt^2}{2}=\tfrac{1}{2}gt^2$$

(c) Discussed along the way:

- Zero time consistent with assumed initial conditions
- Time scale for retarding force is m/γ
- Small time limits sensible – drag force is small – looks like gravity alone
- Large time limits give behavior characteristic of drag force – constant velocity mg/γ and position increasing linearly with time
- Simple equations of motion recovered if drag coefficient goes to zero, for all times.

An accelerometer is made by hanging a simple pendulum of mass m and length L from the top of an otherwise empty box, which accelerates with constant acceleration a in the horizontal direction. Assume that the gravitational acceleration g acts downwards in the vertical direction. Assume further that conditions are arranged so that the pendulum's motion always in the plane defined by a and g (i.e. in the plane of the paper in the diagram below).



(a) Find the equation of motion for the angle θ and find the equilibrium angle θ_e.

(b) Show that the frequency ω of small oscillations of the pendulum about the equilibrium angle θ_e is determined by $\omega^2 = \frac{\sqrt{a^2+g^2}}{L}$.

(c) Briefly discuss the physical significance of the results in (a) and (b).

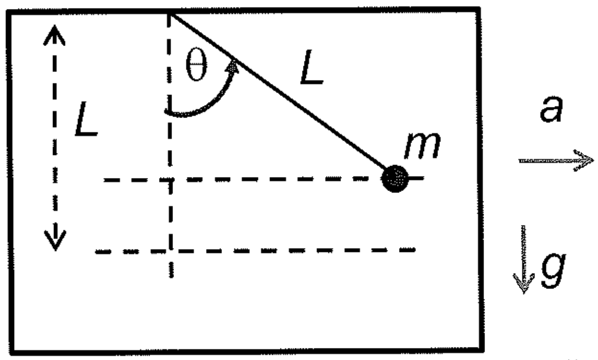
Some of the standard trigonometric identities below may be useful.

$$\begin{aligned}
\cos^2 A + \sin^2 A &= 1 \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B
\end{aligned}$$

$$\begin{aligned}
\sin A &= \pm\sqrt{1-\cos^2 A} = \frac{\pm \tan A}{\sqrt{1+\tan^2 A}} = \frac{e^{iA} - e^{-iA}}{2i} \\
\cos A &= \pm\sqrt{1-\sin^2 A} = \frac{\pm 1}{\sqrt{1+\tan^2 A}} = \frac{e^{iA} + e^{-iA}}{2} \\
\tan A &= \frac{\sin A}{\cos A} = \frac{\pm \sin A}{\sqrt{1-\sin^2 A}} = \frac{\pm\sqrt{1-\cos^2 A}}{\cos A}
\end{aligned}$$

Comprehensive Exam, Winter 2018 CM Grad (Solution)

An accelerometer is made by hanging a simple pendulum of mass m and length L from the top of an otherwise empty box, which accelerates with constant acceleration a in the horizontal direction. Assume that the gravitational acceleration g acts downwards in the vertical direction. Assume further that conditions are arranged so that the pendulum's motion always in the plane defined by a and g (i.e. in the plane of the paper in the diagram below).



Use a Lagrangian approach. Define the origin of the xy coordinate system as the initial position of the point of suspension of the pendulum. x increase to the right; y increases upwards. Let x_0 and v_0 be the initial position and velocity of the box, and though they will turn out not to matter, that is not obvious from the outset and they should be included until they are shown to drop out.

(a) Find the equation of motion for the angle θ and find the equilibrium angle θ .

The motion is that of the uniformly accelerating box plus that of the pendulum bob swinging. The coordinates of the pendulum bob are

$x = x_0 + v_0 t + \tfrac{1}{2}at^2 + L\sin\theta$ and $y = -L\cos\theta$

The velocity coordinates of the pendulum bob are

$\dot{x} = v_0 + at + L\dot{\theta}\cos\theta$ and $y = +L\dot{\theta}\sin\theta$

The <u>potential energy</u>, relative to zero when the pendulum is horizontal, is
$U = mgy = -mgL\cos\theta$.

The <u>kinetic energy</u> is $T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2$.

$$T = \tfrac{1}{2}m\left(v_0 + at + L\dot{\theta}\cos\theta\right)^2 + \tfrac{1}{2}m\left(L\dot{\theta}\sin\theta\right)^2$$

$$= \tfrac{1}{2}m\left(v_0^2 + a^2t^2 + L^2\dot{\theta}^2\cos^2\theta + 2v_0at + 2atL\dot{\theta}\cos\theta + 2v_0L\dot{\theta}\cos\theta\right) + \tfrac{1}{2}mL^2\dot{\theta}^2\sin^2\theta.$$

$$= \tfrac{1}{2}m\underbrace{\left(v_0^2 + a^2t^2 + 2v_0at\right)}_{\text{indep. of }\theta,\,\dot{\theta}} + m\left(at + v_0\right)L\dot{\theta}\cos\theta + \tfrac{1}{2}mL^2\dot{\theta}^2\underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{1}$$

The <u>Lagrangian</u> is

$$L = T - U$$

$$= \tfrac{1}{2}m\left(v_0^2 + a^2t^2 + 2v_0at\right) + m\left(at + v_0\right)L\dot{\theta}\cos\theta + \tfrac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

The Euler-Lagrange equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$.

First term:

$$\frac{\partial L}{\partial \dot{\theta}}=mL^2\dot{\theta}+m\left(at+v_0\right)L\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=mL^2\ddot{\theta}+maL\cos\theta-m\left(at+v_0\right)L\dot{\theta}\sin\theta$$

Second term:

$$\frac{\partial L}{\partial \theta}=\frac{\partial}{\partial \theta}\left(m\left(at+v_0\right)L\dot{\theta}\cos\theta+mgL\cos\theta\right)$$

$$=-m\left(\left(at+v_0\right)\dot{\theta}+g\right)L\sin\theta$$

Plug in to get the equation of motion for the single variable θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0\Rightarrow$$

$$mL^2\ddot{\theta}+maL\cos\theta-\cancel{m\left(at+v_0\right)L\dot{\theta}\sin\theta}+m\left(\cancel{\left(at+v_0\right)\dot{\theta}}+g\right)L\sin\theta=0\Rightarrow$$

$$\boxed{\ddot{\theta}+\frac{a}{L}\cos\theta+\frac{g}{L}\sin\theta=0}$$

At equilibrium, $\ddot{\theta}=0$, which means that the equilibrium angle θ_e is defined by

$$\boxed{\tan\theta_e=-\frac{a}{g}}$$ (see (c) for physics comments)

(b) Show that the frequency ω of small oscillations of the pendulum about the equilibrium angle θ_e is determined by $\omega^2=\frac{\sqrt{a^2+g^2}}{L}$.

Define a **small displacement from equilibrium** $\varphi\equiv\theta-\theta_e$. Note that neither θ nor θ_e is necessarily small – in fact they can be very large. It is φ that is small.

$\theta=\varphi+\theta_e$; $\dot{\varphi}=\dot{\theta}$; $\ddot{\varphi}=\ddot{\theta}$. Use the equation of motion $\ddot{\theta}+\frac{a}{L}\cos\theta+\frac{g}{L}\sin\theta=0$ to get

$$\ddot{\varphi}+\frac{a}{L}\cos\left(\varphi+\theta_e\right)+\frac{g}{L}\sin\left(\varphi+\theta_e\right)=0$$

Use standard trig identities (given):

$$\ddot{\varphi}+\frac{a}{L}\left(\cos\varphi\cos\theta_e-\sin\varphi\sin\theta_e\right)+\frac{g}{L}\left(\sin\varphi\cos\theta_e+\cos\varphi\sin\theta_e\right)=0$$

For small angles $\sin\varphi\approx\varphi$; $\cos\varphi\approx1$

$$\ddot{\varphi} = -\frac{a}{L}(\cos\theta_e - \varphi\sin\theta_e) - \frac{g}{L}(\varphi\cos\theta_e + \sin\theta_e)$$

$$\ddot{\varphi} = -\frac{1}{L}\underbrace{(a\cos\theta_e + g\sin\theta_e)}_{=0 \text{ (see below)}} - \frac{1}{L}\underbrace{(g\cos\theta_e - a\sin\theta_e)}_{\text{constant}}\varphi$$

From the equilibrium angle definition $\tan\theta_e = -\frac{a}{g} \Rightarrow g\sin\theta_e = -a\cos\theta_e$, the first term on the right is indeed zero. Thus we have the form for <u>small angle oscillations</u> about θ_e:

$$\ddot{\varphi} = \underbrace{-\frac{1}{L}(g\cos\theta_e - a\sin\theta_e)}_{\omega^2}\varphi \equiv -\omega^2\varphi$$

It remains to show that the frequency is as advertised, use trig identities (given) to write cos and sin in terms of tan, but you have to argue to choose the +ve sign in front of the root for both sin and cos in terms of tan. If a is in the positive x direction, the pendulum is displaced to the left in equilibrium, which is a negative angle between 0 and $-\pi/2$. In that case the tangent is negative ($-a/g$), the sine is negative and the cosine is positive.

$$\omega^2 L = g\cos\theta_e - a\sin\theta_e = g\frac{+1}{\sqrt{1+\tan^2\theta_e}} - a\frac{+\tan\theta_e}{\sqrt{1+\tan^2\theta_e}} = \frac{g - a\tan\theta_e}{\sqrt{1+\tan^2\theta_e}}$$

$$= \frac{g + a^2/g}{\sqrt{1+a^2/g^2}} = g\frac{1+a^2/g^2}{\sqrt{1+a^2/g^2}} = g\sqrt{1+a^2/g^2} = \sqrt{g^2+a^2}$$

And finally,

$$\boxed{\omega = \frac{\sqrt{a^2+g^2}}{L}}$$

(c) Physics: The equation of motion for θ and equilibrium angle depend only on a, not x_0 and v_0, hence the system is indeed an accelerometer. The bob is displaced left (right) if the acceleration is to the right (left), as seems intuitive. The oscillation frequence has a term that has an "effective gravity" term with a and g added in quadrature. This reflects the perpendicular nature of the two sources of acceleration. As $a \to 0$, the the pendulum hangs vertically, and can exhibit small oscillations about the vertical with the expected frequency $\sqrt{g/L}$. As $a \gg g$, the bob moves to a near horizontal position ($\theta_e \to -\pi/2$) and the frequency of vibration about the equilibrium angle becomes larger as the "effective" gravity increases.

Two parallel conducting plates of area A and separation d_1 ($d_1^2 \ll A$) are connected for a long time to a battery of voltage V and then disconnected. Neglect all edge effects.

(a) How much work is required to separate the plates a small distance $\Delta d \ll d_1$ to the new separation $d_2 = d_1 + \Delta d$?

(b) What force is required to hold the plates apart?

(c) With the separation now d_2, the battery is reconnected. Will any charge flow and, if so, onto or off the plates?

A reminder of equations potentially relevant to this problem: the capacitance and the electric potential energy of the two parallel conducting plates are

$$C = \epsilon_0 \frac{A}{d} \qquad \text{and} \qquad U = \frac{1}{2} C V^2 = \frac{\epsilon_0}{2} \int \vec{E} \cdot \vec{E} \, dr^3 .$$

Two parallel conducting plates of area A and separation d_1 ($d_1^2 \ll A$) are connected for a long time to a battery of voltage V and then disconnected. Neglect all edge effects.

Solution:
Solution (using SI units): Note that the individual parts are independent but also connected. The solution shows several ways but not all possible ones.

(a) How much work is required to separate the plates a small distance $\Delta d \ll d_1$ to the new separation $d_2 = d_1 + \Delta d$?

Solution:
The two plates form an ideal plate capacitor with capacitance $C = \epsilon_0 \frac{A}{d}$ and the energy stored in it is

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{Q^2}{2\epsilon_0 A}d,$$

with the charge determined by the battery voltage V at distance d_1,

$$Q_1 = CV = \epsilon_0 \frac{AV}{d_1}.$$

The work needed to move the plates from distance d_1 to distance $d_2 = d_1 + \Delta d$ is

$$\begin{aligned} W &= \frac{Q_1^2}{2\epsilon_0 A}(d_2 - d_1) \\ &= \frac{\epsilon_0 A V^2}{2d_1^2}(d_2 - d_1). \end{aligned}$$

Note: If part b) was solved first the work is also given by

$$W = \int F dx = F\Delta d = \frac{Q^2}{2\epsilon_0 A}(d_2 - d_1).$$

since the force is independent of d (with Q fixed).

(b) What force is required to hold the plates apart?

Solution:
The two plates form an ideal plate capacitor with capacitance

$$C = \epsilon_0 \frac{A}{d}.$$

The charge on the plates with distance d_1 is

$$Q_1 = CV = \epsilon_0 \frac{AV}{d_1}.$$

The force between the two plates can be found in two ways:

i) determine the force on the charge on one plate in the electric field of the other plate:
The electric field due to the charge on one plate is

$$E = \frac{1}{\epsilon_0}\frac{\sigma}{2} = \frac{1}{\epsilon_0}\frac{Q}{2A}\,.$$

and can be found using Gauss's Law:

$$\int E \cdot dA = \frac{Q}{\epsilon_0}$$
$$2EA = \frac{Q}{\epsilon_0}\,,$$

and

$$E = \frac{Q}{2\epsilon_0 A}\,.$$

The force on a charge q in an electric field E is (magnitudes)

$$\begin{aligned} F &= qE\,, \\ &= \frac{Q_1^2}{2\epsilon_0 A} \\ &= \frac{\epsilon_0 A V^2}{2d_1^2}\,. \end{aligned}$$

ii) determine the force as the negative gradient of the potential energy (energy stored in the capacitor) with charge Q held constant:

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{Q_1^2}{2\epsilon_0 A}d\,,$$

and

$$F = -\frac{\partial U}{\partial d} = -\frac{Q_1^2}{2\epsilon_0 A}\,.$$

The force between the plates is attractive, hence the applied force must hold them apart.

(c) With the separation now d_2, the battery is reconnected. Will any charge flow and, if so, onto or off the plates?

Solution:
The charge on the plates with distance d_2 is

$$Q_2 = CV = \epsilon_0 \frac{AV}{d_2} < Q_1\,,$$

since $d_2 > d_1$. Charge flows from the plates back to the battery.

A reminder of equations potentially relevant to this problem: the capacitance and the electric potential energy of the two parallel conducting plates are

$$C = \epsilon_0 \frac{A}{d} \qquad \text{and} \qquad U = \frac{1}{2} C V^2 = \frac{\epsilon_0}{2} \int \vec{E} \cdot \vec{E} \, dr^3.$$

A electromagnetic plane wave in free space (the free-space electric permittivity and magnetic permeability are $\epsilon = \epsilon_0$ and $\mu = \mu_0$, respectively) falls normally onto a semi-infinite dielectric medium with $\epsilon_1 = 4\epsilon_0, \mu_1 = \mu_0$. Calculate the fraction of the wave's energy that is reflected using the steps outlined below:

(a) Write down the plane wave solutions to Maxwell's equations in the two regions such that the incoming plane wave travels in the positive z direction. Place the interface at $z = 0$. For simplicity (and without loss of generality) choose a fixed polarization. Let the polarization of the $\vec{E}$ field of the incoming plane wave point in the positive x direction.

(b) State the boundary conditions satisfied by the electromagnetic fields at the interface.

(c) Calculate the fraction of the wave's energy that is reflected.

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Solution:
We choose the positive z axis as the direction of propagation of the incident wave and take the direction of polarization of the electric field to point in the positive x direction. The electric and magnetic fields of the incident, transmitted and reflected parts of the wave can then be written as (placing the interface at $z = 0$)

incident wave

$$\vec{E}_I = E_x\hat{e}_x = E_I\hat{e}_x \mathrm{e}^{i(kz-\omega t)}$$
$$\vec{B}_I = B_y\hat{e}_y = B_I\hat{e}_y \mathrm{e}^{i(kz-\omega t)}$$

transmitted wave

$$\vec{E}_T = E_T\hat{e}_x \mathrm{e}^{i(k_1 z-\omega t)}$$
$$\vec{B}_T = B_T\hat{e}_y \mathrm{e}^{i(k_1 z-\omega t)}$$

reflected wave

$$\vec{E}_R = E_R\hat{e}_x \mathrm{e}^{i(-kz-\omega t)}$$
$$\vec{B}_R = B_R\hat{e}_y \mathrm{e}^{i(-kz-\omega t)}$$

We have already made use of the fact that the magnetic field is perpendicular to the direction of propagation and the electric field. The propagation of the reflected wave in the negative z-direction has been made explicit.

(b) State the boundary conditions satisfied by the electromagnetic fields at the interface.

Solution:
Boundary conditions at the interface require continuity for the tangential components for $\vec{E}$

and $\vec{H}$ fields. At the interface ($z = 0$) for the electric field

$$E_I + E_R = E_T \,,$$

and for the magnetic field $B = \mu H$

$$H_I + H_R = H_T$$
$$\frac{1}{\mu} B_I + \frac{1}{\mu} B_R = \frac{1}{\mu_1} B_T$$

Maxwell's equations provide the connection between $\vec{E}$ and $\vec{B}$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \,.$$

We only need the y component of $\vec{B}$:

$$\begin{aligned} \frac{\partial B_y}{\partial t} &= \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \\ &= -\frac{\partial E_x}{\partial z} \,. \end{aligned}$$

For the incident wave we get

$$\frac{\partial B_y}{\partial t} = -ikE_x \,,$$

which we integrate in time to get

$$B_y = \frac{k}{\omega} E_x = \sqrt{\epsilon\mu} E_x \,.$$

Similarly for the transmitted wave

$$B_y = \frac{k_1}{\omega} E_x = \sqrt{\epsilon_1 \mu_1} E_x \,.$$

For the reflected wave (note the sign!)

$$\frac{\partial B_y}{\partial t} = +ikE_x \,,$$

and

$$B_y = -\frac{k}{\omega} E_x = -\sqrt{\epsilon\mu} E_x$$

(c) Calculate the fraction of the wave's energy that is reflected.

Solution:
Putting everything together we can express the b.c. for the magnetic field in terms of the electric field.

$$\sqrt{\frac{\epsilon}{\mu}}E_I - \sqrt{\frac{\epsilon}{\mu}}E_R = \sqrt{\frac{\epsilon_1}{\mu_1}}E_T\,.$$

The energy in the waves is proportional to E^2 and we solve for $(E_R/E_I)^2$:

$$\begin{aligned}\left(\frac{E_R}{E_I}\right)^2 &= \left(\frac{\sqrt{\frac{\epsilon_1}{\mu_1}} - \sqrt{\frac{\epsilon}{\mu}}}{\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon}{\mu}}}\right)^2 \\ &= \left(\frac{n_1 - 1}{n_1 + 1}\right)^2\,,\end{aligned}$$

where the index of refraction of the dielectric medium is

$$n_1 = \sqrt{\frac{\epsilon_1\mu_1}{\epsilon_0\mu_0}}$$

With the numbers provided, $n_1 = 2$ and $(E_R/E_I)^2 = 1/9$.