OSU Physics Department
Comprehensive Examination #129

Wednesday, September 20 and Thursday, September 21, 2017

Fall 2017 Comprehensive Examination

PART 1, Wednesday, September 20, 9:00am
PART 2, Wednesday, September 20, 1:00pm
PART 3, Thursday, September 21, 9:00am
PART 4, Thursday, September 21, 1:00pm

General Instructions

This Fall 2017 Comprehensive Examination consists of four separate parts of two problems each. Each problem caries equal weight (20 points each) and lasts three hours. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit–especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all exams and formula sheets at the end of the exam.

We consider a binary alloy system A_xB_{1-x} made up of N atoms of an element A and $N_0 - N$ atoms of an element B, where $x \equiv N/N_0$. Assume that all the N_0 atomic sites are fixed in space, while any type of atoms can occupy an atomic site.

(a) Find the number of possible arrangements of A and B in the alloy and the entropy associated with these arrangements, which is called the entropy of mixing.

(b) Express the entropy of mixing as a function of x for $N \gg 1$ and $N_0 - N \gg 1$.

(c) A repulsive energy denoted by u_r exists between an A atom and surrounding B atoms. Find the Helmholtz free energy F associated with the mixing in the limit of a very small number of A atoms ($x \ll 1$). Determine the proportion x in thermal equilibrium at temperature T.

You may find useful the Stirling's approximation:

$$\ln n! \cong n \ln n - n \text{ for } n \gg 1$$

We consider a binary alloy system $A_x B_{1-x}$ made up of N atoms of an element A and $N_0 - N$ atoms of an element B, where $x \equiv N/N_0$. Assume that all the N_0 atomic sites are fixed in space, while any type of atoms can occupy an atomic site.

(a) Find the number of possible arrangements of A and B in the alloy and the entropy associated with these arrangements, which is called the entropy of mixing.

Solution:
The number of arrangements of the alloy system is the number of combinations to select N sites out of the N_0 sites:

$$\Omega(N_0, N) = \frac{N_0!}{(N_0 - N)!N!}$$

Therefore, the entropy of mixing is

$$S(N_0, N) = k_B \ln \Omega(N_0, N) = k_B \left[\ln N_0! - \ln(N_0 - N)! - \ln N!\right]$$

(b) Express the entropy of mixing as a function of x for $N \gg 1$ and $N_0 - N \gg 1$.

Solution:
Using the Stirling approximation, we get

$$\begin{aligned} S(N_0, N) &\cong k_B \left[N_0 \ln N_0 - N_0 - (N_0 - N)\ln(N_0 - N) + (N_0 - N) - N \ln N + N\right] \\ &= k_B \left[N_0 \ln N_0 - (N_0 - N)\ln(N_0 - N) - N \ln N\right] \\ &= k_B \left[-(N_0 - N)\ln(1 - N/N_0) - N \ln(N/N_0)\right] \\ \Rightarrow S(x) &= -N_0 k_B \left[(1 - x)\ln(1 - x) + x \ln x\right] \end{aligned}$$

(c) A repulsive energy denoted by u_r exists between an A atom and surrounding B atoms. Find the Helmholtz free energy F associated with the mixing in the limit of a very small number of A atoms ($x \ll 1$). Determine the proportion x in thermal equilibrium at temperature T.

Solution:
For $x \ll 1$, the total repulsive energy is $U = N u_r = x N_0 u_r$. In this limit, the free energy is

$$F(x) = U(x) - TS(x) = N_0 \left\{Ux + k_B T \left[(1 - x)\ln(1 - x) + x \ln x\right]\right\}.$$

The free energy is minimum when

$$\frac{\partial F}{\partial x} = N_0 \left\{U + k_B T \ln\left(\frac{x}{1 - x}\right)\right\} = 0, \text{ thus } \frac{x}{1 - x} = e^{-u_r/k_B T},$$

and, for $x \ll 1$,

$$x = e^{-u_r/k_B T}.$$

You may find useful the Stirling's approximation:

$$\ln n! \cong n \ln n - n \text{ for } n \gg 1$$

A weak magnetic field B is applied to a crystal composed of spin-1/2 particles. There are N particles per unit volume. The magnetic energy of the individual particles is given as

$$\epsilon(s) = -2\mu_0 B s,$$

where $s = \pm 1/2$ is the spin quantum number. The magnetic moment of each particle is either $+\mu_0$ for $s = +1/2$ or $-\mu_0$ for $s = -1/2$.

(a) Find the magnetization M, the magnetic moment per unit volume, at the absolute-zero temperature $T = 0$, using simple physical arguments with no detailed calculation.

(b) Evaluate the magnetization at a finite temperature T. Show that the magnetization in the low temperature limit, $k_B T \ll \mu_0 B$, is consistent with your answer to the part (a). What is the magnetic susceptibility χ at T? Here, we use a more general definition of differential susceptibility, $\chi = \frac{\partial M}{\partial B}$. Find χ for $k_B T \ll \mu_0 B$.

(c) In the limit of an infinitely high temperature, $T \to \infty$, obtain the values of M and χ, using simple physical arguments without any detailed calculation.

(d) Find how $M(T)$ and $\chi(T)$ vary with temperature in the high temperature limit, $k_B T \gg \mu_0 B$. Show that your answer is consistent with your answer to the part (c).

(e) Sketch $\chi(T)$ as a function of T in the entire temperature range.

A weak magnetic field B is applied to a crystal composed of spin-1/2 particles. There are N particles per unit volume. The magnetic energy of the individual particles is given as

$$\epsilon(s) = -2\mu_0 Bs,$$

where $s = \pm 1/2$ is the spin quantum number. The magnetic moment of each particle is either $+\mu_0$ for $s = +1/2$ or $-\mu_0$ for $s = -1/2$.

(a) Find the magnetization M, the magnetic moment per unit volume, at the absolute-zero temperature $T = 0$, using simple physical arguments with no detailed calculation.

Solution:
The system must be in the ground state at $T = 0$. Thus, all the particles are in the lower energy state for $s = +1/2$. Since the magnetic moment of a particle in the ground state is μ_0 and there are N particles per unit volume, the magnetization is

$$M = N\mu_0.$$

(b) Evaluate the magnetization at a finite temperature T. Show that the magnetization in the low temperature limit, $k_B T \ll \mu_0 B$, is consistent with your answer to the part (a). What is the magnetic susceptibility χ at T? Here, we use a more general definition of differential susceptibility, $\chi = \frac{\partial M}{\partial B}$. Find χ for $k_B T \ll \mu_0 B$.

Solution:
The probability of finding the particle in the state of $s = \pm 1/2$ at a finite temperature T is

$$P_\pm = \frac{e^{\pm\beta\mu_0 B}}{e^{\beta\mu_0 B} + e^{-\beta\mu_0 B}}, \text{ where } \beta = 1/k_B T.$$

Thus, the mean magnetic moment of a particle at T is

$$\langle\mu\rangle = \frac{(+\mu_0)e^{\beta\mu_0 B} + (-\mu_0)e^{-\beta\mu_0 B}}{e^{\beta\mu_0 B} + e^{-\beta\mu_0 B}} = \mu_0 \tanh(\beta\mu_0 B).$$

The magnetization is then given by

$$M = N\langle\mu\rangle = N\mu_0 \tanh(\beta\mu_0 B).$$

At low temperatures, $y \equiv \beta\mu_0 B \gg 1$,

$$\tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} \cong \frac{e^y}{e^y} = 1, \text{ then } M \cong N\mu_0,$$

which is consistent with the result in (a).
The magnetic susceptibility is

$$\chi = \frac{\partial M}{\partial B} = N\beta\mu_0^2 \operatorname{sech}^2(\beta\mu_0 B).$$

At low temperatures,

$$\operatorname{sech}^2 y = \frac{4}{(e^y + e^{-y})^2} \cong 4e^{-2y} \cong 0, \text{ thus } \chi \cong \frac{4N\mu_0^2}{k_B T} e^{-\frac{\mu_0 B}{k_B T}} \cong 0.$$

(c) In the limit of an infinitely high temperature, $T \to \infty$, obtain the values of M and χ, using simple physical arguments without any detailed calculation.

Solution:
For $T \to \infty$, the particle spins are randomly oriented and the magnetic energy is negligible compared to the thermal energy. Therefore, $M = 0$ and $\chi = 0$.

(d) Find how $M(T)$ and $\chi(T)$ vary with temperature in the high temperature limit, $k_B T \gg \mu_0 B$. Show that your answer is consistent with your answer to the part (c).

Solution:
In the high temperature limit, $y \equiv \beta\mu_0 B \ll 1$,

$$\tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} \cong \frac{(1+y)-(1-y)}{(1+y)+(1-y)} = y, \text{ thus } M \cong N\beta\mu_0^2 B = \frac{N\mu_0^2 B}{k_B T},$$

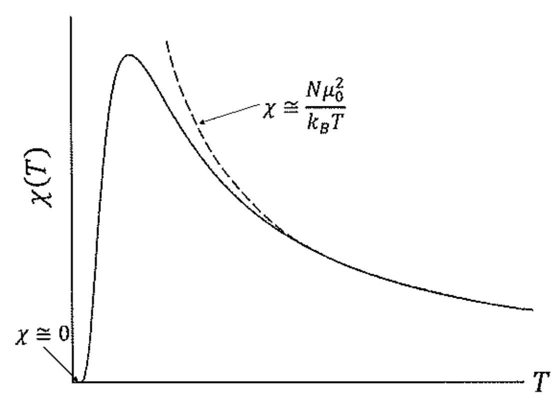
and

$$\operatorname{sech}^2 y = \frac{4}{(e^y + e^{-y})^2} \cong 1, \text{ thus } \chi \cong \frac{N\mu_0^2}{k_B T}.$$

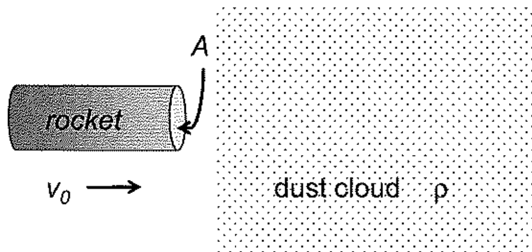
For $T \to \infty$, $M \cong 0$ and $\chi \cong 0$, consistent with the results in (c).

(e) Sketch $\chi(T)$ as a function of T in the entire temperature range.

Solution:
χ is negligible in the low temperature limit and inversely proportional to T in the high temperature limit so that it must have a maximum at a finite temperature.

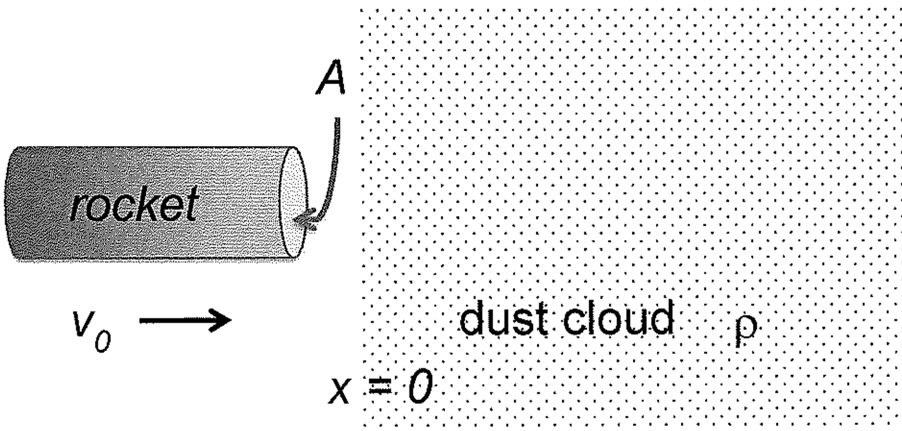


A spacecraft of mass m_0 and cross-sectional area A coasts in free space with velocity v_0, and then encounters a stationary dust cloud of density ρ. Assume that the dust sticks to the craft and that the cross-sectional area does not change with time. Solve for the subsequent motion $x(t)$ of the spacecraft, and show that the limiting case of vanishing density gives the expected result.



Comprehensive Exam, Fall 2017 QM Undergraduate (Solution)

A spacecraft of mass m_0 and cross-sectional area A coasts in free space with velocity v_0, and then encounters a stationary dust cloud of density ρ. Assume that the dust sticks to the craft and that the cross-sectional area does not change with time. Solve for the subsequent motion $x(t)$ of the spacecraft, and show that the limiting case of vanishing density gives the expected result.



The rocket coasts into the dust cloud and accumulates mass from the dust that sticks to the front area as it sweeps through. All dust sticks to the spacecraft so *energy is not conserved* in the resulting completely inelastic collisions, but *conservation of linear momentum* always applies. For this 1-dimensional problem, (i) let m be the mass of the rocket plus the accumulated dust, which obviously increases as a function of time, (ii) let v be the velocity of the rocket as a function of time, and (iii) let x be its position. Choose the time that the rocket encounters the dust to be t=0 so that $x(0)$=0.

Momentum is conserved in a system with no external forces, so

$$\frac{dp}{dt}=0\Rightarrow\frac{d(mv)}{dt}=m\frac{dv}{dt}+\frac{dm}{dt}v=0 \qquad (1)$$

Multiply by dt and separate the variables v and m:

$$\frac{dv}{v}=-\frac{dm}{m}$$

Integrate, noting that the initial conditions are $v_{x=0,t=0}=v_0;\ m_{x=0,t=0}=m_0$

$$\frac{dv}{v}=-\frac{dm}{m}\Rightarrow\int_{v_0}^{v}\frac{dv'}{v'}=-\int_{m_0}^{m}\frac{dm'}{m'} \text{ to get } \qquad \ln\frac{v}{v_0}=-\ln\frac{m}{m_0}=\ln\frac{m_0}{m}$$

so that $\frac{v}{v_0}=\frac{m_0}{m}\Rightarrow mv=m_0v_0$

which is just an explicit statement of conservation of momentum; we could have started from this point (with a careful definition of m as above).

As the rocket moves through a distance dx within the dust, the mass change is given by

$dm = \rho A dx$ and the mass is then $m = m_0 + \rho A x$ (2)

The mass increases linearly with position from its starting value m_0.

Substitute (2) in (1)

$$\left(m_0 + \rho A x\right) v = m_0 v_0$$

Solve for v in terms of x:

$$v = \frac{m_0 v_0}{m_0 + \rho A x}$$

Note that v is the time derivative of x:

$$v = \frac{dx}{dt} = \frac{m_0 v_0}{m_0 + \rho A x}$$

Separate variables: $\left(m_0 + \rho A x\right) dx = m_0 v_0 dt$

Integrate: $\int_{x'=0}^{x} \left(m_0 + \rho A x'\right) dx' = m_0 v_0 \int_{t'=0}^{t} dt'$

Change variables to $u = m_0 + \rho A x;\ du = \rho A dx;\ u(x=0) = m_0$, and note that the clock starts at $t = t_0 = 0$ as the rocket hits the cloud with mass m_0: $\frac{1}{A\rho} \int_{u'=m_0}^{u} u' du' = m_0 v_0 t$

Integrate: $\frac{1}{2} u'^2 \Big|_{m_0}^{u} = A\rho m_0 v_0 t$

$$u^2 = m_0^2 + 2 m_0 v_0 A \rho t$$

Clean up: $u(t) = \sqrt{m_0^2 + 2 m_0 v_0 A \rho t}$

$$m_0 + \rho A x(t) = \sqrt{m_0^2 + 2 m_0 v_0 A \rho t}$$

Final result: $$\boxed{x(t) = -\frac{m_0}{\rho A} + \sqrt{\frac{m_0^2}{\rho^2 A^2} + \frac{2 m_0 v_0}{\rho A} t}}$$

Check limits: $x(t=0) = -\frac{m_0}{\rho A} + \sqrt{\frac{m_0^2}{\rho^2 A^2}} = 0;\quad x(t=\infty) \to \infty \text{ as } \sqrt{t}$

For the case of vanishingly small density, we expect the standard $\boxed{x(t) = v_0 t}$. The derived expression does *not* blow up as might be expected from ρ in the denominator, and is consistent with the expected value as shown below:

First simplify and make dimensionless:

$$x(t) = -\frac{m_0}{\rho A} + \sqrt{\frac{m_0^2}{\rho^2 A^2} + \frac{2m_0 v_0}{A\rho}t}$$

$$= -\frac{m_0}{\rho A} + \sqrt{\frac{m_0^2}{\rho^2 A^2}\left(1 + \frac{2A\rho v_0}{m_0}t\right)}$$

$$= \frac{m_0}{\rho A}\left(-1 + \left(1 + \frac{2A\rho v_0}{m_0}t\right)^{1/2}\right)$$

Expand as a power series in the small quantity $\varepsilon \equiv \frac{2A\rho v_0}{m_0}t$. For long times, the density ρ must be corresponding smaller to qualify as "small".

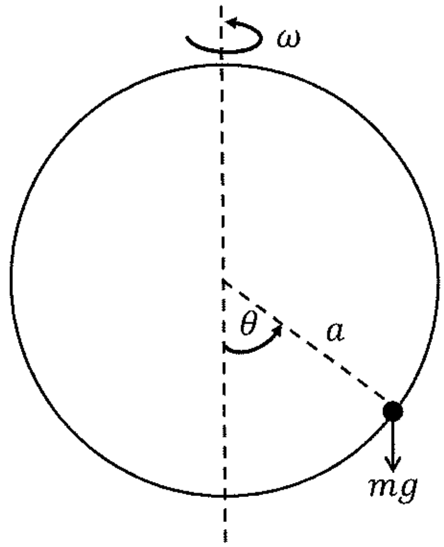
$$x(t) = \frac{m_0}{\rho A}\left(-1 + \left(1 + \frac{2A\rho v_0}{2m_0}t + \left(\frac{1}{2!}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left[\frac{2A\rho v_0}{m_0}\right]^2 t^2 + O\left(\varepsilon^3\right)\right)\right)$$

$$= \frac{m_0}{\not\rho A}\left(\frac{A\not\rho v_0}{m_0}t + \left(\frac{1}{2!}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left[\frac{2A\rho v_0}{m_0}\right]^{2-1} t^2 + O\left(\varepsilon^{3-1}\right)\right)$$

Note that ρ cancels in the first term, and terms after that are proportional to
ρ ρ. So we get the required result to
arbitrary accuracy.

$$x(t) = v_0 t + O(\varepsilon)$$

A bead of mass m slides without friction on a circular loop of radius a. The loop lies in a vertical plane and rotates about a vertical diameter with constant angular velocity ω as shown below.



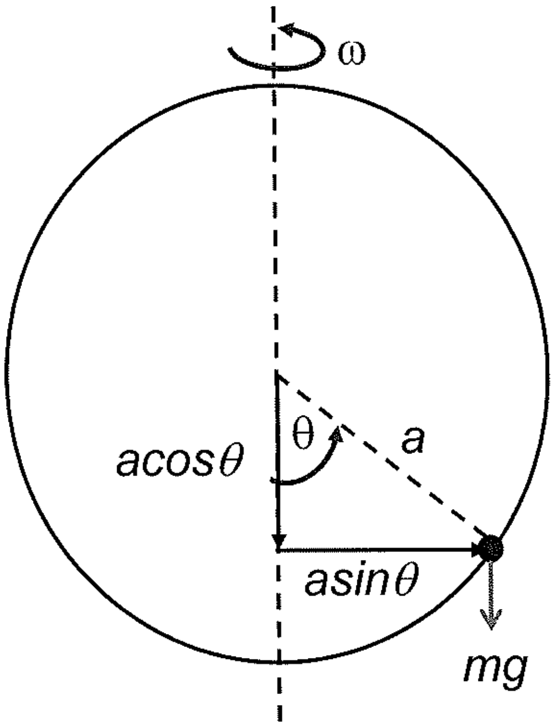
(a) For ω greater than some critical angular velocity, ω_c, the bead can undergo small oscillations about a stable equilibrium point θ_0. Find ω_c and $\theta_0(\omega)$.

(b) Obtain the equations of motion for the small oscillations about θ_0 and find the period of oscillation.

Show your reasoning and algebra carefully.

Comprehensive Exam, Fall 2017 CM Grad (Solution)

A bead of mass m slides without friction on a circular loop of radius a. The loop lies in a vertical plane and rotates about a vertical diameter with constant angular velocity ω as shown below.



(a) For ω greater than some critical angular velocity, ω_c, the bead can undergo small oscillations about a stable equilibrium point θ_0. Find ω_c and $\theta_0(\omega)$.

The potential energy, relative to zero when the bead is horizontal, is $V = -mga\cos\theta$.

The kinetic energy has a term for the (circular) motion along the wire $\frac{1}{2}ma^2\dot{\theta}^2$ and a term for the rotation of the bead around the vertical diameter $\frac{1}{2}m\omega^2(a\sin\theta)^2$. There is no motion in the radial direction because of the constraint $r = a$.

Thus $T = T_\theta + T_\phi = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(a\sin\theta)^2$.

Perhaps a more straightforward method is to note that the bead position and velocity are (with ϕ measured from the x axis in the xy plane; origin at center of circle):

$$x = a\sin\theta\cos\phi; \quad \dot{x} = a\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)$$

$$y = a\sin\theta\sin\phi; \quad \dot{y} = a\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right)$$

$$z = -a\cos\theta; \quad \dot{z} = a\left(\dot{\theta}\sin\theta\right)$$

The kinetic energy is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(a\sin\theta)^2$ (after algebra)

The Lagrangian is $L = T - V = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(a\sin\theta)^2 + mga\cos\theta$

The Euler-Lagrange equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$.

$$\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ma^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = +m\omega^2a^2(\sin\theta\cos\theta) - mga\sin\theta$$

so the equation of motion for the single variable θ is

$$ma^2\ddot{\theta} - m\omega^2 a^2(\sin\theta\cos\theta) + mga\sin\theta = 0$$

$$\ddot{\theta} = \omega^2(\sin\theta\cos\theta) - \frac{g}{a}\sin\theta$$

At equilibrium, $\ddot{\theta} = 0$, which means that the equilibrium angle θ_0 is defined by

$$\boxed{\cos\theta_0 = \frac{g}{a\omega^2}} \qquad \text{(and also } \theta_0 = 0 \text{ (stable) and } \theta_0 = \pi \text{(unstable))}$$

Stable oscillations exist only if $\cos\theta_0 \le 1$ which means there is a critical frequency ω_c

$$\cos\theta_0(\omega_c) = 1 \Rightarrow \frac{g}{a\omega_c^2} = 1 \Rightarrow \omega_c = \sqrt{\frac{g}{a}} \text{ so that } \boxed{\omega > \omega_c; \quad \omega_c = \sqrt{\frac{g}{a}}}$$

(b) Obtain the equations of motion for the small oscillations about θ_0 and find the period of oscillation.

Use Lagrange's equation $a\ddot{\theta} - \omega^2 a(\sin\theta\cos\theta) + g\sin\theta = 0$ and define a **small displacement from equilibrium** $\varphi \equiv \theta - \theta_0$. Note that $\dot{\varphi} = \dot{\theta}; \quad \ddot{\varphi} = \ddot{\theta}; \quad \theta = \varphi + \theta_0$

Then $a\ddot{\theta} - \omega^2 a(\sin\theta\cos\theta) + g\sin\theta = 0$ becomes

$$\ddot{\varphi} - \omega^2\left(\sin(\varphi+\theta_0)\cos(\varphi+\theta_0)\right) + \frac{g}{a}\sin(\varphi+\theta_0) = 0$$

Use standard trig identities:

$$\ddot{\varphi} - \omega^2(\sin\varphi\cos\theta_0 + \cos\varphi\sin\theta_0)(\cos\varphi\cos\theta_0 - \sin\varphi\sin\theta_0) + \frac{g}{a}(\sin\varphi\cos\theta_0 + \cos\varphi\sin\theta_0) = 0$$

For small angles $\sin\varphi \approx \varphi; \quad \cos\varphi \approx 1$

$$\ddot{\varphi} - \omega^2(\varphi\cos\theta_0 + \sin\theta_0)(\cos\theta_0 - \varphi\sin\theta_0) + \frac{g}{a}(\varphi\cos\theta_0 + \sin\theta_0) = 0$$

Use $\cos\theta_0 = \frac{g}{a\omega^2}$, cancel some terms, and neglect terms second order in the small quantity:

$$\ddot{\varphi} - \omega^2\left(\varphi\frac{g}{a\omega^2} + \sin\theta_0\right)\left(\frac{g}{a\omega^2} - \varphi\sin\theta_0\right) + \frac{g}{a}\left(\varphi\frac{g}{a\omega^2} + \sin\theta_0\right) = 0$$

$$\ddot{\varphi} - \omega^2\left(\varphi\left(\frac{g}{a\omega^2}\right)^2 + \frac{g}{a\omega^2}\sin\theta_0 - \underbrace{\varphi^2\frac{g}{a\omega^2}\sin\theta_0}_{\text{second order}} - \varphi\sin^2\theta_0\right) + \frac{g}{a}\left(\varphi\frac{g}{a\omega^2} + \sin\theta_0\right) = 0$$

Clean up:

$$\ddot{\varphi} \approx -\left(\omega^2\sin^2\theta_0\right)\varphi$$

This is the harmonic oscillator equation with a frequency (use $\cos\theta_0 = \frac{g}{a\omega^2}$ again)

$$\boxed{\Omega = \omega\sin\theta_0 = \omega\sqrt{1-\cos^2\theta_0} = \omega\sqrt{1-\frac{g^2}{a^2\omega^4}}}$$

A particle of mass m is confined in a box with a wall in the middle, such that the potential can be expressed as:

$$U(x) = \begin{cases} \infty, x < -a \\ \gamma\delta(x), -a \leq x \leq a, \gamma > 0 \\ \infty, x > a \end{cases}$$

Because of the symmetry of the potential, it is natural to consider parity when solving for energy eigenstates.

(a) for all bound states with odd parity (such that the wave function is an odd function of x), write down the wave functions and energies of the energy eigenstates.

(b) for all bound states with even parity, the energy levels can be expressed as $E^e = \frac{\hbar^2 k^2}{2m}$. Show that k satisfies the transcendental equation $\tan(ka) = -\frac{\hbar^2}{ma\gamma}(ka)$ which can be solved geometrically.

(c) Sketch the ground state energy as a function of γ, and write down the wave function as well as the ground state energy in the limit $\frac{\hbar^2}{ma\gamma} \gg 1$ and $\frac{\hbar^2}{ma\gamma} \ll 1$.

A particle of mass m is confined in a box with a wall in the middle, such that the potential can be expressed as:

$$U(x) = \begin{cases} \infty, x < -a \\ \gamma\delta(x), -a \le x \le a, \gamma > 0 \\ \infty, x > a \end{cases}$$

Because of the symmetry of the potential, it is natural to consider parity when solving for energy eigenstates.

(a) for all bound states with odd parity (such that the wave function is an odd function of x), write down the wave functions and energies of the energy eigenstates.

Solution:
The wave functions of energy eigenstates must satisfy the following conditions:

$$\frac{d^2}{dx^2}\psi + \frac{2m}{\hbar^2}[E - U(x)]\psi = 0 \quad (1)$$

$$\psi(a) = \psi(-a) = 0 \quad (2)$$

$$\psi(0^+) = \psi(0^-) \quad (3)$$

$$\psi'(0^+) - \psi'(0^-) = \frac{2m\gamma}{\hbar^2}\psi(0) \quad (4)$$

where the last equation can be obtained by integrating the Schrödinger-equation near $x = 0$. It is natural to solve the problem by examining the wave functions of the standard particle in a box problems. For odd parities, we find

$$\psi_n^o = \frac{1}{\sqrt{a}}\sin(\frac{n\pi x}{a}) \quad (5)$$

$$E_n^o = \frac{1}{2m}\frac{\pi^2\hbar^2 n^2}{a^2}, n = 1, 2, 3... \quad (6)$$

Therefore these states are not affected by the wall.

(b) for all bound states with even parity, the energy levels can be expressed as $E^e = \frac{\hbar^2 k^2}{2m}$. Show that k satisfies the transcendental equation $\tan(ka) = -\frac{\hbar^2}{ma\gamma}(ka)$ which can be solved geometrically.

Solution:
To satisfy the Schrödinger-equation and even parity, we have to assume the wave function to be $\psi(x) = A\cos kx + B\sin k|x|$. Now applying the boundary condition and continuity equations, we find

$$kB = \frac{m\gamma}{\hbar^2}A \quad (7)$$

$$A\cos ka + B\sin ka = 0 \quad (8)$$

Together, that means

$$\tan ka = -\frac{\hbar^2}{ma\gamma}(ka) \tag{9}$$

This equation can be solved geometrically, by plotting $y = tanx$ and $y = -\frac{\hbar^2}{ma\gamma}x$ on the same axises. Clearly there are multiple solutions and the energies corresponding to the solution is $E^e = \frac{\hbar^2 k^2}{2m}$.

(c) Sketch the ground state energy as a function of γ, and write down the wave function as well as the ground state energy in the limit $\frac{\hbar^2}{ma\gamma} \gg 1$ and $\frac{\hbar^2}{ma\gamma} \ll 1$.

Solution:
Notice that the lowest energy odd parity state has a energy of $E_1^o = \frac{\hbar^2(\frac{\pi}{a})^2}{2m}$. The first thing to do is to compare this with the even parity state energy $E^e = \frac{\hbar^2 k^2}{2m}$.

The solution of equation (9) is such that $\frac{\pi}{2} + n\pi < ka < \pi + n\pi$, and the lowest energy one corresponds to $\frac{\pi}{2} < ka < \pi$. Clearly the even state still is the ground state.

With everything else fixed, when γ is very small, the line $y = -\frac{\hbar^2}{ma\gamma}x$ meets $y = tanx$ at $x = \frac{\pi}{2}$. Increasing γ, the cross point moves to the right. Until when $\gamma \to \infty$, the cross point is $x = \pi$. This allows you to sketch the ground state energy as a function of γ. Notice that in the large γ limit, the energy approaches to the lowest odd parity energy.

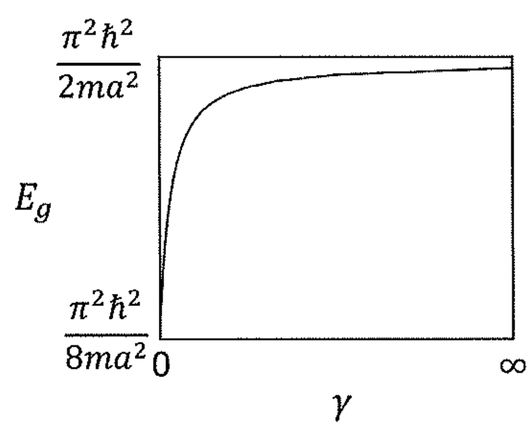


Figure 1: Ground state energy as a function of γ.

Now we write down the wave functions. In the limit $\frac{\hbar^2}{ma\gamma} \gg 1$, we have $ka \approx \pi/2$, and the ground state wave function is approximately $\psi_g(x) = \cos\frac{\pi}{2a}x$ up to a normalization factor.

In the limit $\frac{\hbar^2}{ma\gamma} \ll 1$, we have $ka \approx \pi$, and the ground state wave function is approximately $\psi_g(x) = \sin\frac{\pi}{a}|x|$. No wonder this state has the same energy as the first odd parity state.

Two quantum point particles of equal masses m interact through a harmonic potential $U(r) = \frac{k}{2}r^2$, where r is the relative distance between the two particles. These two particles are considered to be distinguishable. The motion of the particles is confined to a 2-D x-y plane, and their center of mass is at rest.

(a) Write down the ground state wave function and energy of the system. What is the mean square distance between the two particles at ground state?

(b) The first excited state is degenerate. What is the energy of the state? Find two energy eigenstates (in terms of wave functions) that are mutually orthogonal and belong to the first excited state.

(c) What is L_z (the angular momentum along z-axis) of each of the orthogonal states you have found in (b)? Clearly indicate if your answer corresponds to eigenvalues of L_z, or expectation values.

Note: The wave functions of the ground state and first excited state of a 1D harmonic oscillator are

$$\psi_0(x) = \frac{\sqrt{\alpha}}{\pi^{\frac{1}{4}}}\exp[-\frac{1}{2}\alpha^2x^2]$$
$$\psi_1(x) = \frac{\sqrt{2\alpha}}{\pi^{\frac{1}{4}}}\alpha x\exp[-\frac{1}{2}\alpha^2x^2]$$

where $\alpha = \sqrt{\frac{m\omega_0}{\hbar}}$, $\omega_0 = \sqrt{\frac{k}{m}}$.

Two quantum point particles of equal masses m interact through a harmonic potential $U(r) = \frac{k}{2}r^2$, where r is the relative distance between the two particles. These two particles are considered to be distinguishable. The motion of the particles is confined to a 2-D x-y plane, and their center of mass is at rest.

(a) Write down the ground state wave function and energy of the system. What is the mean square distance between the two particles at ground state?

Solution:
Because the center of mass is not moving, we only need to consider the relative motion, which is effectively a 2D harmonic oscillator with a reduced mass $\mu = m/2$. The ground state $|\psi_g = \psi_0(x)\psi_0(y)\rangle$ wave function is

$$\psi_g = \psi_0(x)\psi_0(y) = \frac{\alpha}{\pi^{\frac{1}{2}}}\exp[-\frac{1}{2}\alpha^2\rho^2] \tag{10}$$

where x and y are the relative position of the two particles in Cartesian coordinates. ρ is their relative distance. The ground state energy is $\hbar\omega$, where $\omega = \sqrt{\frac{2k}{m}}$.

To find the mean square distance $\langle\rho^2\rangle$, we note that

$$\begin{aligned} \langle\rho^2\rangle &= \langle\psi_g|x^2|\psi_g\rangle + \langle\psi_g|y^2|\psi_g\rangle \\ &= 2\langle\psi_0(x)|x^2|\psi_0(x)\rangle \\ &= 1/\alpha^2 \end{aligned} \tag{11}$$

The mean square distance between the particles is $1/\alpha^2$.

(b) The first excited state is degenerate. What is the energy of the state? Find two energy eigenstates (in terms of wave functions) that are mutually orthogonal and belong to the first excited state.

Solution:
The first excited state is a space spanned by two basis. One such choice is $\psi_0(x)\psi_1(y)$ and $\psi_1(x)\psi_0(y)$. The energy of the first excited state is $2\hbar\omega$, $\omega = \sqrt{\frac{2k}{m}}$.

(c) What is L_z (the angular momentum along z-axis) of each of the orthogonal states you have found in (b)? Clearly indicate if your answer corresponds to eigenvalues of L_z, or expectation values.

Solution:
Intuitively, $\psi_0(x)\psi_1(y)$ and $\psi_1(x)\psi_0(y)$ correspond to oscillations in the y and x directions respectively. Such linear oscillations have zero angular momentum. However, special linearly superpositions of $\psi_0(x)\psi_1(y)$ and $\psi_1(x)\psi_0(y)$ give rise to circular motions that do have angular momentum.

Given this intuition, we first construct L_z eigen states out of $\psi_0(x)\psi_1(y)$ and $\psi_1(x)\psi_0(y)$. We find such eigen states of L_z are $\psi_\pm = \psi_1(x)\psi_0(y) \pm i\psi_0(x)\psi_1(y) = e^{\pm i\phi}F(\rho)$, where ϕ is the relative polar angle, and ρ is the relative distance between the two particles. It is clear that $\psi_\pm$ have $\pm\hbar$ angular momentum L_z.

Therefore, $\psi_0(x)\psi_1(y)$ and $\psi_1(x)\psi_0(y)$ are not eigen states of L_z, and they give zero expectation values of L_z.

Note: the wave functions of the ground state and first excited state of a 1D harmonic oscillator are:

$$\begin{aligned}\psi_0(x) &= \frac{\sqrt{\alpha}}{\pi^{\frac{1}{4}}}\exp[-\frac{1}{2}\alpha^2x^2]\\ \psi_1(x) &= \frac{\sqrt{2\alpha}}{\pi^{\frac{1}{4}}}\alpha x\exp[-\frac{1}{2}\alpha^2x^2]\end{aligned}$$

where $\alpha = \sqrt{\frac{m\omega_0}{\hbar}}$, $\omega_0 = \sqrt{\frac{k}{m}}$.

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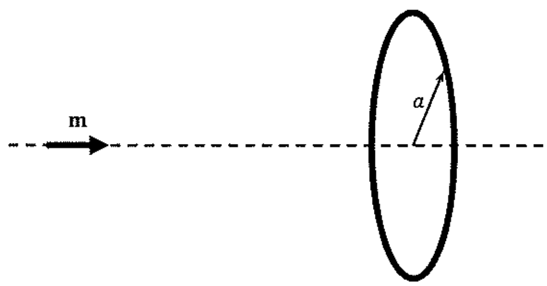
where $\alpha = \sqrt{\frac{m\omega_0}{\hbar}}$, $\omega_0 = \sqrt{\frac{k}{m}}$.

A fixed circular loop (radius a) is oriented in the y-z plane and centered on the x-axis. An ideal (point) magnetic dipole moment $\mathbf{m} = m\hat{x}$ is located on the x-axis a distance x ($x \gg a$) away from the center of the loop. The magnetic dipole moment points towards the loop as indicated in the figure. Its dimensions are much less than x or a.



(a) The loop initially carries a current I in the counter clockwise direction when the loop is viewed from the dipole. Calculate the change in potential energy of the dipole if the direction of the dipole is reversed. Express your answer in terms of I, a, x and m.

(b) The current in the loop is now turned off. The magnetic moment points again towards the loop. You move the dipole on the x-axis towards the loop at a constant velocity v ($v \ll c$). At all times you maintain $x \gg a$. To keep the velocity constant, what is the force you must apply to counter the magnetic force? Neglect the self-inductance of the loop. Express your answer in terms of m, a, v, x and R, where R is the resistance of the loop.

(c) Explain qualitatively how your answer to part (b) would change if self-inductance is not negligible. Discuss the limits under which the self-inductance, L, can be neglected.

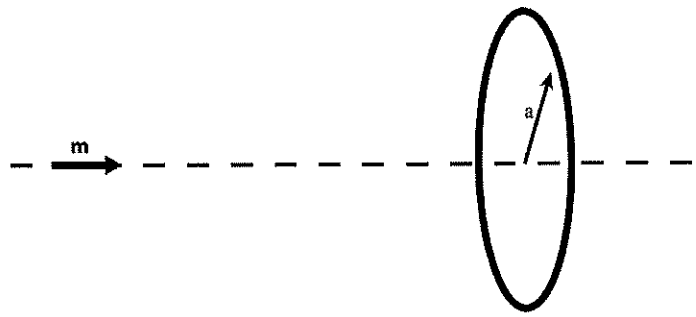
The magnetic field $\mathbf{B}(\mathbf{r})$ due to a current I in a loop C' is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{\mathbf{dr}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad (SI)$$

$$\mathbf{B}(\mathbf{r}) = \frac{I}{c} \oint_{C'} \frac{\mathbf{dr}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad (Gaussian)$$

7. Undergraduate level E&M

A fixed circular loop (radius a) is oriented in the y-z plane and centered on the x-axis. An ideal (point) magnetic dipole moment $\mathbf{m} = m\hat{x}$ is located on the x-axis a distance x ($x \gg a$) away from the center of the loop. The magnetic dipole moment points towards the loop as indicated in the figure. Its dimensions are much less than x or a.



1. The loop initially carries a current I in the counter clockwise direction when the loop is viewed from the dipole. Calculate the change in potential energy of the dipole if the direction of the dipole is reversed. Express your answer in terms of I,a,x and m.

   The magnetic field of the current carrying ring is calculated using the law of Biot-Savart (given). Axial symmetry ensures that on axis the only non zero component of the B field points in the x direction.

$$\begin{aligned} B\hat{\mathbf{e}}_x &= \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{\mathbf{dr}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \\ &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{-a^2 d\theta \hat{\mathbf{e}}_x}{(x^2+a^2)^{3/2}} \\ &= \frac{-\mu_0 I a^2}{2(x^2+a^2)^{3/2}}\hat{\mathbf{e}}_x \\ &\approx \frac{-\mu_0 I a^2}{2x^3}\hat{\mathbf{e}}_x \qquad (x \gg a) \end{aligned} \tag{1}$$

   The potential energy of a magnetic dipole in a magnetic field is $U = -\vec{m}\cdot\vec{B}$. The B field from the current carrying loop points in the negative x direction, hence initially the magnetic dipole points in the opposite direction of the magnetic field.

$$\begin{aligned} \Delta U &= U_{\text{reversed}} - U_{\text{initial}} = (-m|B|) - (+m|B|) = \\ &= -2m|B| = \frac{-\mu_0 I a^2 m}{x^3} \end{aligned} \tag{2}$$

2. The current in the loop is now turned off. The magnetic moment points again towards the loop. You move the dipole on the x-axis towards the loop at a constant velocity v ($v \ll c$). At all times you maintain $x \gg a$. To keep the velocity constant, what is the force you must apply to counter the magnetic force? Neglect the self-inductance of the loop. Express your answer in terms of m, a, v, x and R, where R is the resistance of the loop.

   The magnetic field of the dipole at the loop can be assumed constant over the extend of the loop ($2a$) since $x \gg a$. The field of a magnetic dipole is given by (may be taken from formula sheet)

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}$$

On axis the field points in the x direction (axial symmetry) and is given by

$$\begin{aligned} B\hat{\mathbf{e}}_x &= \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{e}}_x(m) - m\hat{\mathbf{e}}_x}{r^3} \\ &= \frac{\mu_0 m}{2\pi x^3} \hat{\mathbf{e}}_x \end{aligned} \tag{3}$$

The magnitude of the induced current in the loop is given by Faraday's law

$$I = \frac{1}{R} \frac{d\Phi}{dt},$$

where the magnetic flux Φ is given by

$$\Phi = BA = \frac{\mu_0 a^2 m}{2x^3},$$

where A is the area of the loop. We obtain

$$I = \frac{1}{R} \frac{d\Phi}{dt} = \frac{3\mu_0 a^2 m v}{2x^4 R}. \tag{4}$$

where $v = dx/dt$. The induced current will generate a magnetic field at the position of the dipole as calculated in the first part, with the direction of the field counteracting the field of the magnetic dipole (Lenz's law). The field of the magnetic dipole points in the positive x direction at the position of the loop, hence the induced magnetic field points in the negative x direction and the induced current flows counter clockwise as in the first part.

$$B\hat{\mathbf{e}}_x = -\frac{3\mu_0^2 a^4 m v}{4x^7 R} \hat{\mathbf{e}}_x. \tag{5}$$

Finally the force acting on ther moving dipole due to the induced current in the loop is

$$F = -\frac{\partial U}{\partial x} = -\frac{\partial(-mB)}{\partial x} = -\frac{21\mu_0^2 a^4 m^2 v}{4x^8 R}. \tag{6}$$

You must apply a force of equal magnitude in the opposite direction (the positive x direction) to keep the velocity constant.

3. Explain qualitatively how your answer to part (b) would change if self-inductance is not negligible. Discuss the limits under which the self-inductance, L, can be neglected.

Including self-induction, we have

$$\frac{d\Phi}{dt} = IR + L\frac{dI}{dt}. \tag{7}$$

Self-induction can be ignored if

$$L\frac{dI}{dt} \ll IR, \tag{8}$$

where we assume absolute values everywhere, since we only compare magnitudes. Using the result for the current I obtained in the previous part, we can estimate

$$\frac{dI}{dt} = \frac{dI}{dx}\frac{dx}{dt} = \frac{4}{x}Iv, \tag{9}$$

and

$$\begin{aligned} L\frac{4v}{x}I &\ll IR, \\ L &\ll \frac{Rx}{4v}. \end{aligned} \tag{10}$$

Self-inductance can be ignored if the dipole is far away and/or moves slowly and/or if the electrical resistance of the loop is large.

The magnetic field $\mathbf{B}(\mathbf{r})$ due to a current I in a loop C' is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_{C'} \frac{\mathbf{dr}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad (SI)$$

$$\mathbf{B}(\mathbf{r}) = \frac{I}{c}\oint_{C'} \frac{\mathbf{dr}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad (Gaussian)$$

A point charge q is placed a distance d from the center of a homogeneous dielectric sphere (dielectric permittivity ϵ) of radius R.

(a) Calculate the electric potential everywhere, i.e inside and outside the sphere.

(b) For large distances, $d \gg R$, calculate the force on the point charge due to the electric field of the polarized sphere. Interpret your result.

8. Graduate level E&M

A point charge q is placed a distance d from the center of a homogeneous dielectric sphere (dielectric permittivity ϵ) of radius R.

1. Calculate the electric potential everywhere, i.e inside and outside the sphere.

The problem has axial symmetry and the potential satisfies Laplace's equation, which can be solved using an expansion in Legendre polynomials in 3 separate regions:

$$\begin{aligned} 1: \quad r &\leq R \\ 2: \quad R &\leq r < d \\ 3: \quad r &> d \end{aligned}$$

The contribution from the sphere and the point charge must be included separately. The expansion in Legendre polynominals in the general case

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

and for a point charge

$$\frac{1}{|\mathbf{x} - r'\hat{z}|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

are given in the formula sheet.

For our problem, note that in region 1 terms $\sim \frac{b_l}{r^{l+1}}$ must vanish ($b_l = 0$) to avoid divergences at the origin and in region 3 terms $\sim a_l r^l$ must vanish ($a_l = 0$) to avoid divergences at ∞. Region 2 and 3 must have the same expansion for the contribution to the potential from the dielectric sphere to ensure that the potential is continuous at the boundary $r = d$. We get:

$$\phi_1(r,\theta) = \sum_l a_l r^l P_l(\cos\theta) + \sum_l \frac{qr^l}{d^{l+1}} P_l(\cos\theta), \qquad r \leq R \qquad (1)$$

$$\phi_2(r,\theta) = \sum_l \frac{b_l}{r^{l+1}} P_l(\cos\theta) + \sum_l \frac{qr^l}{d^{l+1}} P_l(\cos\theta), \qquad R \leq r < d \qquad (2)$$

$$\phi_2(r,\theta) = \sum_l \frac{b_l}{r^{l+1}} P_l(\cos\theta) + \sum_l \frac{qd^l}{r^{l+1}} P_l(\cos\theta), \qquad r > d. \qquad (3)$$

The remaining coefficients a_l and b_l are determinded by the boundary conditions for ϕ and $\mathbf{D}$ at the surface of the sphere, $r = R$. Uniqueness of the Legendre polynomial expansion allows us to equate terms for each l seperately. The continuity of ϕ gives

$$a_l R^l = \frac{b_l}{R^{l+1}}, \qquad (4)$$

$$b_l = a_l R^{2l+1}. \qquad (5)$$

The continuity of D_n at $r = R$ written in terms of the potential ϕ is

$$\epsilon \left. \frac{\partial \phi_1(r,\theta)}{\partial r} \right|_{r=R} = \left. \frac{\partial \phi_2(r,\theta)}{\partial r} \right|_{r=R}. \tag{6}$$

This gives

$$\epsilon l a_l R^{l-1} + \frac{\epsilon l q R^{l-1}}{d^{l+1}} = \frac{-(l+1)b_l}{R^{l+2}} + \frac{l q R^{l-1}}{d^{l+1}}. \tag{7}$$

Finally we can solve for a_l

$$a_l = \frac{(1-\epsilon) l q}{(\epsilon l + l + 1) d^{l+1}} \tag{8}$$

and b_l

$$b_l = \frac{(1-\epsilon) l q R^{2l+1}}{(\epsilon l + l + 1) d^{l+1}}. \tag{9}$$

The potential is given by Eqs. (1)-(3) with coefficients given in Eqs. (8) and (9).

2. For large distances, $d \gg R$, calculate the force on the point charge due to the electric field of the polarized sphere. Interpret your result.

The force is calculated form the potential using only the terms derived form the sphere

$$\begin{aligned} F_q &= -q \left. \frac{\partial \phi_{\text{sphere}}(r,0)}{\partial r} \right|_{r=d}, \\ &= \sum_l \frac{l(l+1)(1-\epsilon)q^2}{(\epsilon l + l + 1)d^2} \left(\frac{R}{d}\right)^{2l+1}. \end{aligned} \tag{10}$$

At large distances, the $l = 1$ dipole term dominates and the force is given by

$$F_q = \left(\frac{1-\epsilon}{\epsilon+2}\right) \frac{2q^2R^3}{d^5}. \tag{11}$$

This is the force between a point charge and an induced dipole.

Not required: the induced dipole is given by

$$\mathbf{p} = \left(\frac{1-\epsilon}{\epsilon+2}\right) \frac{qR^3}{d^2}. \tag{12}$$

The force between a point charge and a dipole falls of like $1/d^3$ but we see here that the force between a point charge and an induced dipole falls of like $1/d^5$ because the induced dipole has a $1/d^2$ dependence.