OSU Physics Department
Comprehensive Examination #116

Thursday, April 4 and Friday, April 5, 2013

Spring 2013 Comprehensive Examination

PART 1, Thursday, April 4, 9:00am

General Instructions

This Spring 2013 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Thursday, April 4, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Friday, April 5, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for “scratch” work, separated by at least one empty page from your solutions. “Scratch” work will not be graded.

The energy levels of a system are given by $E_n = n\epsilon$, $n = 1, 2, \cdots$, where ϵ is a constant. The degeneracy of the n-th level is $g_n = n^{D-1}$, where D is an integer with typical values of 1,2,or 3. Calculate the internal energy for this system for D=1 and D=2. What are the expressions for large temperature values? Calculate the high temperature limit of the heat capacity in both cases.

The energy levels of a system are given by $E_n = n\epsilon$, $n = 1, 2, \cdots$, where ϵ is a constant. The degeneracy of the n-th level is $g_n = n^{D-1}$, where D is an integer with typical values of 1,2,or 3. Calculate the internal energy for this system for D=1 and D=2. What are the expressions for large temperature values? Calculate the high temperature limit of the heat capacity in both cases.

Solution:

$$\mathcal{Z} = \sum_{n=1}^{\infty} n^{D-1} e^{-n\frac{\epsilon}{k_B T}}$$

Use the abbreviation

$$F_P(x) = \sum_{n=1}^{\infty} n^P x^n$$

and we see that

$$\mathcal{Z} = F_{D-1}(e^{-\frac{\epsilon}{k_B T}})$$

We know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

and hence

$$F_0(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

Also,

$$\frac{d}{dx} F_P(x) = \sum_{n=1}^{\infty} n^P n x^{n-1} = \frac{1}{x} F_{P+1}(x)$$

which gives us

$$F_1(x) = x\frac{d}{dx}F_0(x) = x\left(\frac{1}{1-x} + \frac{x}{(1-x)^2}\right) = \frac{x}{1-x} + \frac{x^2}{(1-x)^2} = \frac{x}{(1-x)^2}$$

$$F_2(x) = x\frac{d}{dx}F_1(x) = x\left(\frac{1}{(1-x)^2} + 2\frac{x}{(1-x)^3}\right) = \frac{x+x^2}{(1-x)^3}$$

The Helmholtz free energy follows from

$$F(T) = -k_B T \ln(\mathcal{Z}) = -k_B T \ln(F_{D-1}(e^{-\frac{\epsilon}{k_B T}}))$$

The entropy is given by

$$S = -\left(\frac{\partial F}{\partial T}\right) = -\frac{F(T)}{T} + k_B T \frac{1}{F_{D-1}(e^{-\frac{\epsilon}{k_B T}})} F'_{D-1}(e^{-\frac{\epsilon}{k_B T}}) e^{-\frac{\epsilon}{k_B T}} \frac{\epsilon}{k_B T^2}$$

We have $U = F + TS$ and hence

$$U(T) = \epsilon \frac{1}{F_{D-1}(e^{-\frac{\epsilon}{k_B T}})} F'_{D-1}(e^{-\frac{\epsilon}{k_B T}}) e^{-\frac{\epsilon}{k_B T}} = \epsilon \frac{F_D(e^{-\frac{\epsilon}{k_B T}})}{F_{D-1}(e^{-\frac{\epsilon}{k_B T}})}$$

This gives for $D = 1$:

$$\frac{F_1(e^{-\frac{\epsilon}{k_B T}})}{F_0(e^{-\frac{\epsilon}{k_B T}})} = \frac{1}{1 - e^{-\frac{\epsilon}{k_B T}}}$$

$$U(T) = \epsilon \frac{1}{1 - e^{-\frac{\epsilon}{k_B T}}}$$

The high temperature limit is

$$U(T) \approx \epsilon \frac{1}{1 - [1 - \frac{\epsilon}{k_B T}]} = k_B T$$

This gives for $D = 2$:

$$\frac{F_2(e^{-\frac{\epsilon}{k_B T}})}{F_1(e^{-\frac{\epsilon}{k_B T}})} = \frac{1 + e^{-\frac{\epsilon}{k_B T}}}{1 - e^{-\frac{\epsilon}{k_B T}}}$$

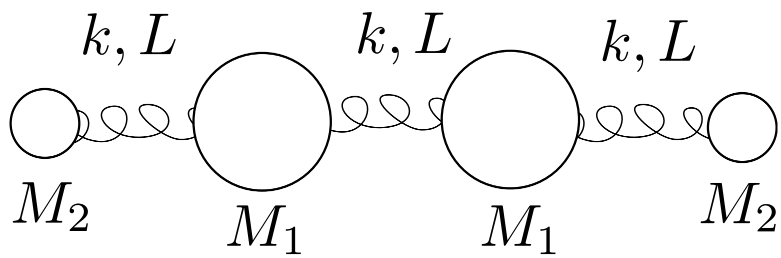
$$U(T) = \epsilon \frac{1 + e^{-\frac{\epsilon}{k_B T}}}{1 - e^{-\frac{\epsilon}{k_B T}}}$$

The high temperature limit is

$$U(T) \approx \epsilon \frac{2}{1 - [1 - \frac{\epsilon}{k_B T}]} = 2 k_B T$$

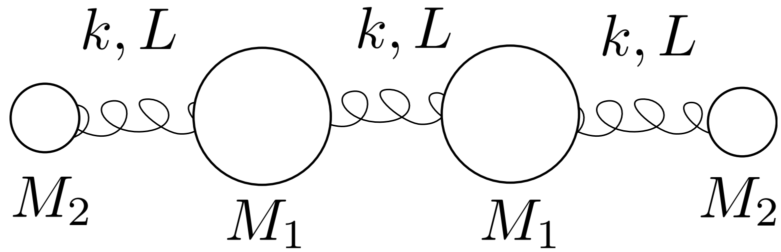
The heat capacities are now easy. We have for $D = 1$ $C = k_B$ and for $D = 2$ we have $C = 2k_B$.

Consider the following one-dimensional arrangement of masses and springs. There are four masses attached by three identical springs. The inner two masses are identical, as are the outer pair of masses.



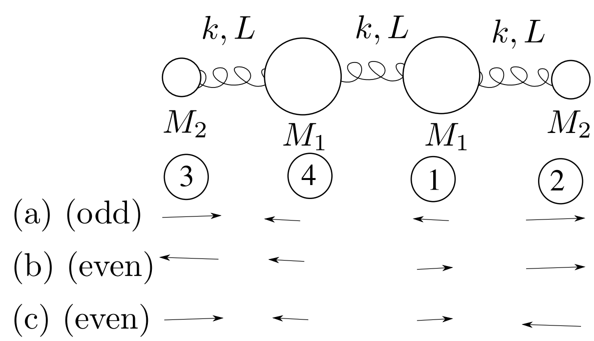
Solve for the normal modes and their frequencies. Please sketch the normal modes. You only need consider motion in one dimension (the horizontal one), and when specifying the normal modes you need only provide the *relative* amplitude of motion for each coordinate.

Consider the following one-dimensional arrangement of masses and springs. There are four masses attached by three identical springs. The inner two masses are identical, as are the outer pair of masses.



Solve for the normal modes and their frequencies. Please sketch the normal modes. You only need consider motion in one dimension (the horizontal one), and when specifying the normal modes you need only provide the *relative* amplitude of motion for each coordinate.

Solution:
Because the system itself is symmetric, we know that we will see only symmetric and antisymmetric solutions (even and odd[1]). Moreover, we can see that there will be three normal modes, since there are four degrees of freedom (the x position of each mass) and there is one translational motion. Using the symmetry of the system, we can see that there should be one odd mode (the overall translational is also odd) and two even modes, which I sketch below, and label (a), (b) and (c). In each case the two identical masses either have the same displacement or opposite displacement, so there are only two variables needed to describe each mode.



Mode (a) We will begin with mode (a), which is the odd mode. In this mode, the middle spring never exerts a force, since the two M_1 masses have identical

[1] In the case of phonon modes, odd and even can be confusing. In this solution I define as even the mode that preserves the symmetry of the system, while the odd mode is the one that breaks it. If you disagree with this designation, that is all right.

motion. Therefore, the center of mass of each of the two side pairs must be fixed, which tells us that

$$M_1 x_1 = -M_2 x_2 \tag{1}$$
$$x_1 = -\frac{M_2}{M_1} x_2 \tag{2}$$

Thus we already know the eigenvector using conservation of momentum. We should still check, which we can do by writing down the two equations of motion, which we can do while solving for the frequency.

$$\begin{aligned}
M_1 \ddot{x}_1 &= -k(x_1 - x_2) && (3)\\
&= -M_1 \omega^2 x_1 && (4)\\
\omega^2 x_1 &= \frac{k}{M_1}(x_1 - x_2) && (5)\\
M_2 \ddot{x}_2 &= -k(x_2 - x_1) && (6)\\
&= -M_2 \omega^2 x_2 && (7)\\
\omega^2 x_2 &= -\frac{k}{M_2}(x_1 - x_2) && (8)\\
&= -\frac{M_1}{M_2}\omega^2 x_1 \quad \text{We did have the right eigenvector...} && (9)\\
\omega^2 x_2 &= -\frac{k}{M_2}\left(-\frac{M_2}{M_1}x_2 - x_2\right) && (10)\\
&= k\left(\frac{1}{M_1} + \frac{1}{M_2}\right)x_2 && (11)
\end{aligned}$$

So our final result for normal mode (a) is:

$$\boxed{\omega_{(a)} = \sqrt{k\left(\frac{1}{M_1} + \frac{1}{M_2}\right)}}\boxed{x_1^{(a)} = 1}\boxed{x_2^{(a)} = -\frac{M_1}{M_2}}\boxed{x_3^{(a)} = -\frac{M_1}{M_2}}\boxed{x_4^{(a)} = 1} \tag{12}$$

I should note here that I am numbering the masses as labelled above so 1 and 4 are of mass M_1 while 2 and 3 are of mass M_2.

Mode (b) and (c) We have to treat modes (b) and (c) together, since they have the same symmetry. As before, we will begin by writing down the equations of motion, making use of the symmetry. In this case, the force due to the middle spring is non-zero.

$$\begin{aligned}
M_2 \ddot{x}_2 &= -k(x_2 - x_1) && (13)\\
&= -M_2 \omega^2 x_2 && (14)\\
M_1 \ddot{x}_1 &= -k(x_1 - x_2) - k(2x_1) && (15)\\
&= -M_1 \omega^2 x_1 && (16)
\end{aligned}$$

At this point we have a bonna fide 2×2 eigenvalue problem to solve. There are more approaches to solve this than I can shake a stick at. I began by writing the problem in matrix form.

$$\begin{pmatrix} 3\frac{k}{M_1} & -\frac{k}{M_1} \\ -\frac{k}{M_2} & \frac{k}{M_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \omega^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} 3M_2 & -M_2 \\ -M_1 & M_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\omega^2 M_1 M_2}{k} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} 3M_2 & -M_2 \\ -M_1 & M_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{19}$$

In the last line I defined a convenient eigenvalue the incorporates some of the constants. So now we've written things as a matrix equation, it's easy to solve the characteristic equation.

$$\begin{vmatrix} 3M_2 - \lambda & -M_2 \\ -M_1 & M_1 - \lambda \end{vmatrix} = 0 \tag{20}$$

$$0 = (3M_2 - \lambda)(M_1 - \lambda) - M_1 M_2 \tag{21}$$

$$= \lambda^2 - (3M_2 + M_1)\lambda + 2M_1 M_2 \tag{22}$$

$$\lambda = \frac{3M_2 + M_1 \pm \sqrt{(3M_2 + M_1)^2 - 8M_1 M_2}}{2} \tag{23}$$

$$\omega^2 = \frac{k}{2M_1 M_2} \left(3M_2 + M_1 \pm \sqrt{(3M_2 + M_1)^2 - 8M_1 M_2} \right) \tag{24}$$

This gives us eigenvalues, but we still don't have the eigenvectors. For that we'll want to go back to our eigenvalue equations:

$$\begin{pmatrix} 3M_2 & -M_2 \\ -M_1 & M_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{25}$$

$$-M_1 x_1 + M_1 x_2 = \lambda x_2 \tag{26}$$

$$x_1 = \left(1 - \frac{\lambda}{M_1} \right) x_2 \tag{27}$$

$$= \left(1 - \frac{3M_2 + M_1 \pm \sqrt{(3M_2 + M_1)^2 - 8M_1 M_2}}{2M_1} \right) x_2 \tag{28}$$

$$= \left(\frac{1}{2} - \frac{3M_2 \pm \sqrt{(3M_2 + M_1)^2 - 8M_1 M_2}}{2M_1} \right) x_2 \tag{29}$$

So this is it. Putting (b) and (c) together, we have:

$$\boxed{\omega_{(b,c)} = \sqrt{\frac{k}{2M_1M_2}\left(3M_2 + M_1 \pm \sqrt{(3M_2+M_1)^2 - 8M_1M_2}\right)}} \tag{30}$$

or equivalently

$$\boxed{\omega_{(b,c)} = \sqrt{\frac{k}{2}\left(\frac{3}{M_1} + \frac{1}{M_2} \pm \sqrt{\frac{9}{M_1^2} + \frac{1}{M_2^2} - \frac{2}{M_1M_2}}\right)}} \tag{31}$$

$$\boxed{x_1^{(b,c)} = \left(\frac{1}{2} - \frac{3M_2 \pm \sqrt{(3M_2+M_1)^2 - 8M_1M_2}}{2M_1}\right)} \tag{32}$$

$$\boxed{x_2^{(b,c)} = 1} \tag{33}$$

$$\boxed{x_3^{(b,c)} = -1} \tag{34}$$

$$\boxed{x_4^{(b,c)} = -\left(\frac{1}{2} - \frac{3M_2 \pm \sqrt{(3M_2+M_1)^2 - 8M_1M_2}}{2M_1}\right)} \tag{35}$$

So there you have it, frequencies and non-normalized eigenvectors.

A simple Hilbert space is defined by an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$. The vectors (kets) $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are defined in terms of this basis

$$|\Phi_1\rangle = N_1 \left(|1\rangle + \frac{1}{2}|2\rangle - i|3\rangle \right) \qquad \text{and} \qquad |\Phi_2\rangle = N_2 \left(\frac{1}{\sqrt{2}}|1\rangle + \frac{i}{\sqrt{2}}|3\rangle \right) .$$

(a) Normalize $|\Phi_1\rangle$ and $|\Phi_2\rangle$. Determine a vector $|\Phi_3\rangle$ such that $\{|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle\}$ form an orthonormal basis.

(b) Find a matrix representation in the basis $\{|1\rangle, |2\rangle, |3\rangle\}$ of the projection operators that project onto the vectors $|\Phi_1\rangle$, $|\Phi_2\rangle$, and $|\Phi_3\rangle$. Verify that these matrices are Hermitian and satisfy the completeness relation.

(c) The Hamiltonian of a system in this Hilbert space is given by

$$\hat{H} = \begin{pmatrix} -E_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_0 \end{pmatrix}$$

in a matrix representation with respect to the basis $\{|1\rangle, |2\rangle, |3\rangle\}$. Determine the time evolution of the vectors $|\Phi_i\rangle$ in this system. What energies do you measure for vector $|\Phi_1(t)\rangle$, and with what probabilities?

$$|\phi_1\rangle = N_1\left(|1\rangle + \tfrac{1}{2}|2\rangle - i|3\rangle\right)$$
$$|\phi_2\rangle = N_2\left(1/\sqrt{2}\,|1\rangle + \tfrac{i}{\sqrt{2}}|3\rangle\right)$$

a) $$\langle\phi_1|\phi_1\rangle = |N_1|^2\left(\langle 1| + \tfrac{1}{2}\langle 2| + i\langle 3|\right)\left(|1\rangle + \tfrac{1}{2}|2\rangle - i|3\rangle\right)$$
$$= |N_1|^2\left(1 + \tfrac{1}{4} + 1\right) = |N_1|^2\,\tfrac{9}{4}$$

$$\Rightarrow \underline{\underline{N_1 = \tfrac{2}{3}}}$$

$$\langle\phi_2|\phi_2\rangle = |N_2|^2\left(\tfrac{1}{2} + \tfrac{1}{2}\right) = |N_2|^2 \Rightarrow \underline{\underline{N_2 = 1}}$$

Verify $$\langle\phi_1|\phi_2\rangle = N_1^* N_2\left(\langle 1| + \tfrac{1}{2}\langle 2| + i\langle 3|\right)\left(\tfrac{1}{\sqrt{2}}|1\rangle + \tfrac{i}{\sqrt{2}}|3\rangle\right)$$
$$= N_1^* N_2\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right) = 0 \quad \checkmark$$

With $|\phi_3\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$

$$\langle\phi_1|\phi_3\rangle = N_1^*\left(\langle 1| + \tfrac{1}{2}\langle 2| + i\langle 3|\right)\left(c_1|1\rangle + c_2|2\rangle + c_3|3\rangle\right)$$
$$= N_1^*\left(c_1 + \tfrac{1}{2}c_2 + ic_3\right) \overset{!}{=} 0 \qquad (1)$$
$$\langle\phi_2|\phi_3\rangle = \dots = N_2^*\left(\tfrac{1}{\sqrt{2}}c_1 - \tfrac{i}{\sqrt{2}}c_3\right) \overset{!}{=} 0 \qquad (2)$$

(2) $\Rightarrow c_3 = -ic_1$, in (1) $\Rightarrow 2c_1 + \tfrac{1}{2}c_2 = 0$
$\Leftrightarrow c_2 = -4c_1$

$$\langle\phi_3|\phi_3\rangle = |c_1|^2\left(\langle 1| - 4\langle 2| + i\langle 3|\right)\left(|1\rangle - 4|2\rangle - i|3\rangle\right)$$
$$= |c_1|^2\left(1 + 16 + 1\right) \Rightarrow c_1 = \frac{1}{\sqrt{18}}$$

$$\Rightarrow |\phi_3\rangle = \frac{1}{\sqrt{18}}\left(|1\rangle - 4|2\rangle - i|3\rangle\right)$$

b) projectors $\hat{P}_i = |\phi_i\rangle\langle\phi_i|$

$$\hat{P}_1 = |\phi_1\rangle\langle\phi_1| = \frac{4}{9}\begin{pmatrix} 1 \\ \frac{1}{2} \\ -i \end{pmatrix}\left(1, \frac{1}{2}, i\right) = \frac{4}{9}\begin{pmatrix} 1 & \frac{1}{2} & i \\ \frac{1}{2} & \frac{1}{4} & \frac{i}{2} \\ -i & -\frac{i}{2} & 1 \end{pmatrix}$$

$(\hat{P}_1)_{ij} = (\hat{P}_1)_{ji}^* \Rightarrow$ Hermitian

$$\hat{P}_2 = |\phi_2\rangle\langle\phi_2| = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}(1, 0, -i) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{pmatrix}$$

$(\hat{P}_2)_{ij} = (\hat{P}_2)_{ji}^* \Rightarrow$ Hermitian

$$\hat{P}_3 = |\phi_3\rangle\langle\phi_3| = \frac{1}{18}\begin{pmatrix} 1 \\ -4 \\ -i \end{pmatrix}(1, -4, i) = \frac{1}{18}\begin{pmatrix} 1 & -4 & i \\ -4 & 16 & -4i \\ -i & 4i & 1 \end{pmatrix}$$

$(\hat{P}_3)_{ij} = (\hat{P}_3)_{ji}^* \Rightarrow$ Hermitian

completeness relation:

$$\sum_{i=1}^{3} |\phi_i\rangle\langle\phi_i| = \frac{1}{18}\begin{pmatrix} 8+9+1 & 4+0-4 & 8i-9i+i \\ 4+0-4 & 2+0+16 & 4i+0-4i \\ -8i+9i-i & -4i+0+4i & 8+9+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$$

$\square$

c) $\hat{H}|1\rangle = -E_0|1\rangle$, $\hat{H}|2\rangle = 0\cdot|2\rangle$, $\hat{H}|3\rangle = E_0|3\rangle$

$\Rightarrow |1\rangle, |2\rangle, |3\rangle$ are energy eigenstates.

time evolution:

$$|\phi_j(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\phi_j\rangle$$

$$|\phi_1(t)\rangle = \frac{2}{3}\left(e^{i\frac{E_0}{\hbar}t}|1\rangle + \frac{1}{2}|2\rangle - i\,e^{-i\frac{E_0}{\hbar}t}|3\rangle\right)$$

$$|\phi_2(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{i\frac{E_0}{\hbar}t}|1\rangle + i\,e^{-i\frac{E_0}{\hbar}t}|3\rangle\right)$$

$$|\phi_3(t)\rangle = \frac{1}{\sqrt{18}}\left(e^{i\frac{E_0}{\hbar}t}|1\rangle - 4|2\rangle - i\,e^{-i\frac{E_0}{\hbar}t}|3\rangle\right)$$

measure energies $-E_0, 0, E_0$ with probabilities

$$p(-E_0) = |\langle 1|\phi_1(t)\rangle|^2$$

$$= \frac{4}{9}\left|e^{i\frac{E_0}{\hbar}t}\langle 1|1\rangle\right|^2 = \frac{4}{9}$$

$$p(0) = \quad = \frac{1}{9}$$

$$p(E_0) = \quad = \frac{4}{9}$$

Verify $\sum_E p(E) = \dots = 1$ ✓

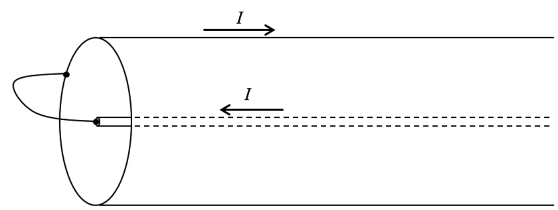
Note: The probabilities are just the diagonal elements of $\hat{\rho}_1$.

A coaxial cable consists of a solid, cylindrical inner conductor with radius r_1, and a hollow cylindrical outer conductor with inside radius of r_2. Both cylinders share the same axis.

Part 1: Capacitance per unit length of a coaxial cable

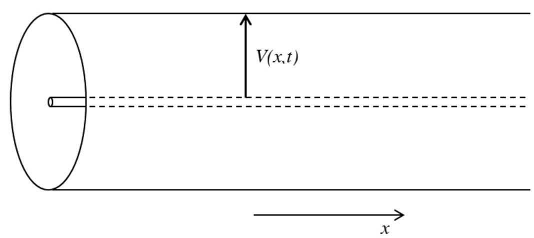
Assume there is a static charge per unit length, λ, on the inner conductor and an equal and opposite charge per unit length -λ on the outer conductor. Calculate the voltage difference between the inner and outer conductors, and hence find the capacitance per unit length, C_0.

Part 2: Inductance per unit length of a coaxial cable



Assume there is a steady current, I, flowing in the inner and outer conductors, as shown above. Calculate the magnetic field generated by this current, and hence the inductance per unit length, L_0. Note that inductance is related to the energy stored in the magnetic field, $U = ½\, L\, I^2$.

Part 3: Voltage disturbance in a coaxial cable



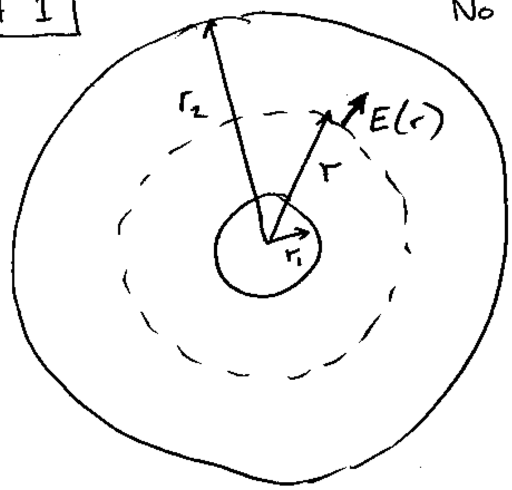
An infinitely long coaxial cable is initialized with zero voltage difference between the inner and outer conductors. Then at $t = 0$ a local voltage disturbance is introduced (non-zero voltage in a short segment of the cable). The time evolution of the disturbance is given by

$$\frac{\partial^2 V(x,t)}{\partial x^2} = L_0 C_0 \frac{\partial^2 V(x,t)}{\partial t^2}$$

- Use the differential equation to find the speed that this voltage disturbance will move along the cable. Your "proof" should include traveling wave solutions to the differential equation.
- Express the propagation speed in terms of r_1, r_2 and fundamental constants.
- Simplify your expression for propagation speed so that it includes as few variables as possible.

$\boxed{\text{Part 1}}$

No field outside



$$2\pi r\, E(r)\, \ell = \frac{Q}{\epsilon_0}$$

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$V = \int_{r_1}^{r_2} E(r)\, dr$$

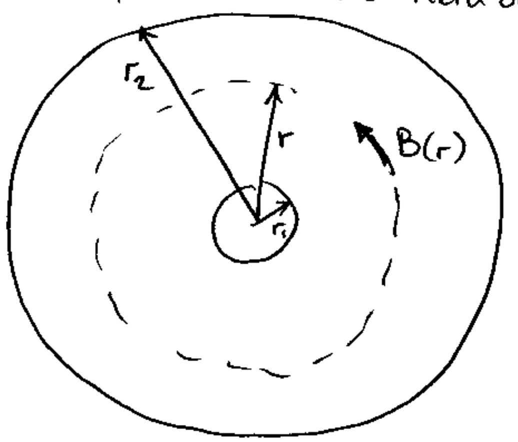
$$= \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

$$C = \frac{Q}{V}$$

$$C_0 = \frac{\lambda}{V} = \frac{\lambda}{\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_2}{r_1}\right)} = \frac{2\pi\epsilon_0}{\ln\left(\frac{r_2}{r_1}\right)}$$

$\boxed{\text{Part 2}}$

No field outside



$$2\pi r B = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

Energy stored in B-field

$$U = \frac{1}{2\mu_0}\int B^2\, dV$$

$$\mathcal{U} = \frac{l}{2\mu_0} \int_{r_1}^{r_2} \frac{\mu_0^2 I^2}{(2\pi)^2 r^2} \, 2\pi r \, dr$$

$$= \frac{l}{2\mu_0} \frac{\mu_0^2 I^2}{2\pi} \int_{r_1}^{r_2} \frac{1}{r} \, dr$$

$$= \frac{l}{2} \frac{\mu_0 I^2}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

We know that $\mathcal{U}$ is also given by

$$\mathcal{U} = \frac{1}{2} L I^2$$

$$\Rightarrow \quad L = \frac{\mu_0}{2\pi} \ln\left(\frac{r_2}{r_1}\right) l$$

and $\boxed{L_0 = \frac{\mu_0}{2\pi} \ln\left(\frac{r_2}{r_1}\right)}$

Part 3 Guess the PDE will be satisfied by a traveling wave solution $V(x,t) = V_0 \sin(kx - \omega t)$.

Arbitrary waveforms could be created by superimposing these Fourier components.

$$\frac{\partial^2 V}{\partial x^2} = -k^2 V_0 \sin(kx - \omega t) \qquad \frac{\partial^2 V}{\partial t^2} = -\omega^2 V_0 \sin(kx - \omega t)$$

Plug into PDE

$$\Longrightarrow \quad k^2 = L_0 C_0 \omega^2$$

i.e. PDE is satisfied when $\frac{\omega}{k} = \frac{1}{\sqrt{L_0 C_0}}$

The phase velocity of a traveling wave is $\frac{\omega}{k}$.

$\Rightarrow$ Disturbances propagate at a velocity

$$v = \frac{1}{\sqrt{L_0 C_0}}$$

$$= \left(\frac{1}{\frac{\mu_0}{2\pi} \ln\left(\frac{r_2}{r_1}\right) \frac{2\pi \epsilon_0}{\ln\left(\frac{r_2}{r_1}\right)}} \right)^{1/2}$$

$$= \left(\frac{1}{\mu_0 \epsilon_0} \right)^{-1/2}$$

This is equal to the speed of light, c.

Consider a particle (mass m) in a one-dimensional potential well

$$V_0(x) = \begin{cases} 0 & |x| \leq a \\ \infty & |x| > a\,. \end{cases}$$

Determine the energy of the ground state (lowest energy eigenstate) for the potential $V(x) = V_0(x) + U(x)$ in lowest non-vanishing order of perturbation theory for the following perturbation potentials:

(a) $U(x) = U_0 \cos(\pi x/a)$,

(b) $U(x) = U_0 \delta(x - b)$, with $-a < b < a$,

(c) $U(x) = U_0 x$.

Hint: $$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^5} = \frac{15\pi^2 - \pi^4}{3 \cdot 16^3}.$$

particle, mass m, in 1D,

potential $V(x) = V_0(x) + U(x)$

with $V_0(x) = \begin{cases} 0 & |x| \le a \\ \infty & |x| > a \end{cases}$

solutions for $U(x) = 0$ (guess or simply calc)

even sols.

$$\psi_n^g(x) = \frac{1}{\sqrt{a}} \cos\left((2n+1)\frac{\pi x}{2a}\right)$$

$$E_n^g = \frac{\hbar^2}{2m}(2n+1)^2\left(\frac{\pi}{2a}\right)^2 \qquad n = 0, 1, 2, \ldots$$

odd sols.

$$\psi_n^u(x) = \frac{1}{\sqrt{a}} \sin\left(n\frac{\pi x}{a}\right)$$

$$E_n^u = \frac{\hbar^2}{2m} n^2 \left(\frac{\pi}{a}\right)^2 \qquad n = 1, 2, 3, \ldots$$

$\rightarrow$ ground state has even symmetry.

a) $U(x) = U_0 \cos\left(\frac{\pi x}{a}\right)$,

$$E_0^{(1)} = \langle \psi_0^g | U(x) | \psi_0^g \rangle$$

$$= \frac{1}{a}\int_{-a}^{a} dx \cos\left(\frac{\pi x}{2a}\right) U_0 \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{2a}\right)$$

$$= \frac{U_0}{a}\int_{-a}^{a} dx \cos\left(\frac{\pi x}{a}\right) \underbrace{\cos^2\left(\frac{\pi x}{2a}\right)}_{=\frac{1}{2}\left(\cos\left(\frac{\pi x}{a}\right)+1\right)}$$

$$= \frac{U_0}{2a}\int_{-a}^{a} dx \cos\left(\frac{\pi x}{a}\right) + \frac{U_0}{2a}\int_{-a}^{a} dx \cos^2\left(\frac{\pi x}{a}\right)$$

$$= \quad \emptyset \quad + \frac{U_0}{4a}\int_{-a}^{a} dx \left[\left(\cos\frac{2\pi x}{a}\right) + 1\right]$$

$$= \qquad \emptyset + \frac{U_0}{2} = \frac{U_0}{2}$$

1st order PT is sufficient:

$$E = E_0^g + E_0^{(1)} = \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2 + \frac{U_0}{2}$$

b) $U(x) = U_0\,\delta(x-b)$, $-a < b < a$

$$E_0^{(1)} = \langle \psi_0 | U | \psi_0 \rangle = \frac{1}{a}\int_{-a}^{a} dx \cos^2\left(\frac{\pi}{2a}x\right)\delta(x-b)$$

$$= \frac{U_0}{a}\cos^2\left(\frac{\pi}{2}\frac{b}{a}\right)$$

$$E = E_0^g + E_0^{(1)}$$

c) $U(x) = U_0 x$ has odd symmetry ! $U^u(x)$

$E_0^{(1)} = \langle \psi_0^g | U^u | \psi_0^g \rangle = 0$ because overall symmetry is odd and $\int_{-a}^{a} f^u(x)\,dx = 0$

$\Rightarrow$ 2nd order PT: $E_0^{(2)} = \sum_{m \neq 0} \frac{|\langle 0|U|m\rangle|^2}{E_0 - E_m}$

matrix elements:

$\langle \psi_0^g | U | \psi_n^g \rangle = 0$ because overall symmetry is odd.
$n = 1, 2, 3, \ldots$

$$\langle \psi_0^g | U | \psi_n^u \rangle = \frac{1}{a}\int_{-a}^{a} dx \cos\left(\frac{\pi x}{2a}\right) U_0 x \sin\left(\frac{\pi}{a}nx\right)$$
$n = 1, 2, 3, \ldots$

from formula sheet:
$2\sin x\cos y = \sin(x+y) + \sin(x-y)$

$= \ldots \longrightarrow$

$$\%_0 = \frac{U_0}{2a}\int_{-a}^{a} dx\, x\left(\sin\left(\frac{\pi}{a}n x + \frac{\pi}{2a}x\right) + \sin\left(\frac{\pi}{a}n x - \frac{\pi}{2a}x\right)\right)$$

$$= \frac{U}{2a}\int_{-a}^{a} dx\; x \sin\left(\frac{\pi}{2a}(2n+1)x\right) + \frac{U}{2a}\int_{-a}^{a} dx\; x \sin\left(\frac{\pi}{2a}(2n-1)x\right)$$

partial integration:

$$\int_{-a}^{a} dx\; x\cdot\sin(c_\pm x) \qquad \text{with } c_\pm = \frac{\pi}{2a}(2n\pm 1)$$

$$= \left[-\frac{x}{c_\pm}\cos(c_\pm x)\right]_{-a}^{a} - (-1)\int_{-a}^{a} dx\; \frac{1}{c_\pm}\cos(c_\pm x)$$

$$= -2\frac{a}{c_\pm}\cos(c_\pm a) + \frac{1}{c_\pm^2}\Big[\sin(c_\pm x)\Big]_{-a}^{a}$$

$$= -\frac{2a}{c_\pm}\cos(c_\pm a) + \frac{2}{c_\pm^2}\sin(c_\pm a)$$

$$\cos(c_\pm a) = \cos\left(\frac{\pi}{2}(2n\pm 1)\right) = \cos\left(n\pi \pm \frac{\pi}{2}\right) = 0$$

$$\sin(c_\pm a) = \sin(\qquad\qquad) = \sin\left(n\pi \pm \frac{\pi}{2}\right) = \pm(-1)^n$$

$$\langle \psi_0^g | U | \psi_n^u \rangle = \frac{U_0}{a}\frac{1}{c_+^2}\sin(c_+ a) + \frac{U_0}{a}\frac{1}{c_-^2}\sin(c_- a)$$

$$= \frac{4U_0 a}{\pi^2}\left(\frac{(-1)^n}{(2n+1)^2} - \frac{(-1)^n}{(2n-1)^2}\right)$$

$$= \frac{4U_0 a}{\pi^2}(-1)^n\left(\frac{(2n-1)^2 - (2n+1)^2}{(4n^2-1)^2}\right)$$

$\longrightarrow$

$$\% = \frac{4U_0 a}{\pi^2}(-1)^n \frac{-8n}{(4n^2-1)^2}$$

$$|\langle \psi_0^g | U | \psi_n^u \rangle|^2 = \frac{16 U_0^2 a^2}{\pi^4} \cdot \frac{64 n^2}{(4n^2-1)^4}$$

$$E_0^g - E_n^u = \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2 - \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 n^2$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2 \left(1 - 4n^2\right)$$

$$E_0^{(2)} = \frac{(16 U_0 a)^2}{\pi^4} \cdot \frac{2m}{\hbar^2} \frac{4^2 a^2}{\pi^2} \underbrace{\sum_{n=1}^{\infty} \frac{-n^2}{(4n^2-1)^5}}_{= -\frac{15\pi^2 - \pi^4}{3 \cdot 16^3}}$$

$$E_0^{(2)} = \frac{2 U_0^2 a^4 m}{3 \pi^4 \hbar^2}(\pi^2 - 15) \qquad < 0$$

$$E \cong E_0^g + E_0^{(2)}$$

The Helmholtz free energy of an ideal gas is given by

$$F_{id}(T,V,N) = NRT\left(\ln(\frac{n}{n_Q(T)}) - 1\right)$$

where $n = \frac{N}{V}$ and $n_Q(T)$ has the dimensions of density and is proportional to $T^{\frac{3}{2}}$. R is the molar gas constant.

(a) Suppose we have a gas with an equation of state given by

$$\frac{pV}{NRT} = 1 + B_2(T)\frac{N}{V}$$

Calculate the difference ΔF in Helmholtz free energy between this gas and the ideal gas by analyzing the work done in an isothermal process.

(b) The coefficient of thermal expansion is given by

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,N}$$

Calculate the coefficient of thermal expansion for the ideal gas and for our gas with the equation of state above.

(c) Suppose that in a certain range of values for the state variables we find that $\Delta F \propto T^2$. What can you say about the coefficient of thermal expansion of our gas compared to the ideal gas?

The Helmholtz free energy of an ideal gas is given by

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Solution:
The first law tells us that

$$dF = -SdT - pdV + \mu dN$$

and hence at constant temperature

$$F(T,V_2,N) - F(T,V_1,N) = -\int_1^2 pdV = -\int_1^2 \frac{NRT}{V}(1 + B_2(T)\frac{N}{V})dV$$

For the ideal gas we have

$$F_{id}(T,V_2,N) - F_{id}(T,V_1,N) = -\int_1^2 \frac{NRT}{V}dV$$

The difference ΔF in Helmholtz free energy then follows from

$$\Delta F(T,V_2,N) - \Delta F(T,V_1,N) = -B_2(T)\int_1^2 \frac{N^2RT}{V^2}dV = B_2(T)N^2RT\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

We also see that for very large values of the volume our gas behaves like an ideal gas. So we take the limit $V_1 \to \infty$ and in this limit the difference in Helmholtz free energy is zero. Hence

$$\Delta F(T,V,N) = B_2(T)N^2RT\frac{1}{V}$$

(b) The coefficient of thermal expansion is given by

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,N}$$

Calculate the coefficient of thermal expansion for the ideal gas and for our gas with the equation of state above.

Solution:
From the equation of state we see, with N constant,

$$dp = \frac{NR}{V}\left(1 + B_2(T)\frac{N}{V}\right)dT + \frac{NRT}{V}B_2'(T)\frac{N}{V}dT + \left(-\frac{NRT}{V^2} - 2B_2(T)\frac{N^2RT}{V^3}\right)dV$$

with p constant we find

$$\frac{NR}{V}\left(1 + B_2(T)\frac{N}{V}\right)dT + \frac{NRT}{V}B_2'(T)\frac{N}{V}dT = \left(\frac{NRT}{V^2} + 2B_2(T)\frac{N^2RT}{V^3}\right)dV$$

and hence

$$\alpha = \frac{1}{V}\frac{\frac{NR}{V}\left(1 + B_2(T)\frac{N}{V}\right) + \frac{NRT}{V}B_2'(T)\frac{N}{V}}{\frac{NRT}{V^2} + 2B_2(T)\frac{N^2RT}{V^3}}$$

Using the density $n = \frac{N}{V}$ we can simplify this somewhat

$$\alpha = \frac{1}{T}\frac{1 + B_2(T)n + nTB_2'(T)}{1 + 2B_2(T)n}$$

The result for the ideal gas follows by setting B_2 identical to zero, which leaves $\frac{1}{T}$.

(c) Suppose that in a certain range of values for the state variables we find that $\Delta F \propto T^2$. What can you say about the coefficient of thermal expansion of our gas compared to the ideal gas?

Solution:
If we have $\Delta F \propto T^2$ we see that in that case

$$B_2(T) = cT$$

and we have

$$B_2'(T) = c$$

Hence

$$\alpha = \frac{1}{T}\frac{1 + cTn + nTc}{1 + 2cTn} = \frac{1}{T}$$

Hence in that particular case there is no change in the thermal expansion coefficient!

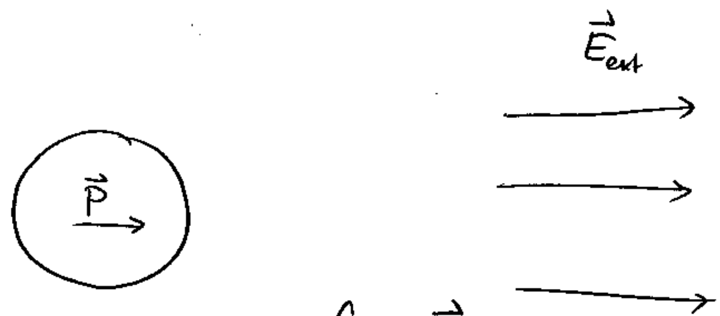
In the absence of additional electric fields, the electric field inside a sphere of uniform polarization **P** is

$$\mathbf{E} = -\frac{1}{3\varepsilon_0}\mathbf{P}$$

a) Use the above relationship to calculate the polarization of a dielectric sphere that is placed in a uniform external electric field, $\mathbf{E}_{\text{ext}}$. Assume that the dielectric sphere is made of a linear polarizable material with electric susceptibility $\chi < 1$.

b) When the dielectric sphere is resting at ground level ($z = 0$), it can be lifted against the force of gravity by applying a non-zero $\mathbf{E}_{\text{ext}}(z = 0)$ and non-zero $d\mathbf{E}/dz|_{z=0}$. Assuming that **E** always points in the z-direction (vertical direction), find the condition for levitation in terms of E, dE/dz, χ, the acceleration due to gravity, g, and the mass density of the sphere, ρ.

a)



The dielectric sphere feels $\vec{E}_{ext}$ as well as the field generated by its own polarization.

$$\vec{P} = \epsilon_0 \chi E_{tot}$$

$$= \epsilon_0 \chi \left(\vec{E}_{ext} - \frac{1}{3\epsilon_0} \vec{P} \right)$$

$$P = \epsilon_0 \chi \left(E_{ext} - \frac{P}{3\epsilon_0} \right)$$

$$P = \epsilon_0 \chi E_{ext} - \frac{\chi}{3} P$$

$$P \left(1 + \frac{\chi}{3} \right) = \epsilon_0 \chi E_{ext}$$

$$\boxed{\vec{P} = \frac{\epsilon_0 \chi}{1 + \chi/3} \vec{E}_{ext}}$$

b) Net dipole moment is $\vec{d} = \vec{P}V$
where V is volume of sphere.

Electric Force on the dipole is $\vec{F}_{elec} = \frac{d\vec{E}}{dz} \cdot \vec{d}$

$$F_{elec,z} = \frac{dE}{dz} PV$$

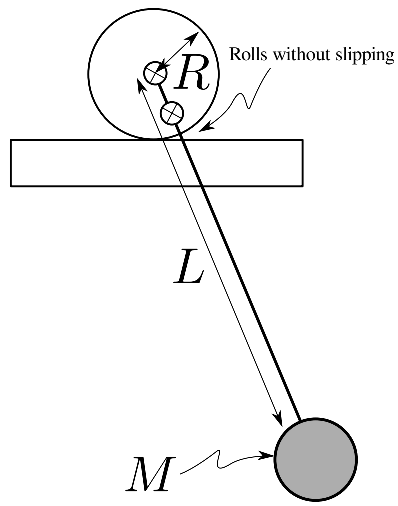
Downward force is $F_{grav,g} = -\rho V g$

$$F_{elec,z} > -F_{grav,g}$$

$$\frac{dE}{dz} P > \rho g$$

$$\boxed{\frac{dE}{dz}\left(\frac{\epsilon_0 \chi}{1+\frac{\chi}{3}}\right)E > \rho g}$$

Consider the following pendulum. It consists of a rigid pendulum that is rigidly attached to a wheel that rolls without slipping. You may neglect the mass of everything but the mass at the end of the pendulum.



(a) Find the equation of motion for this system.

(b) Solve for the period of oscillation, as a function of the amplitude. Your solution may contain an integral.

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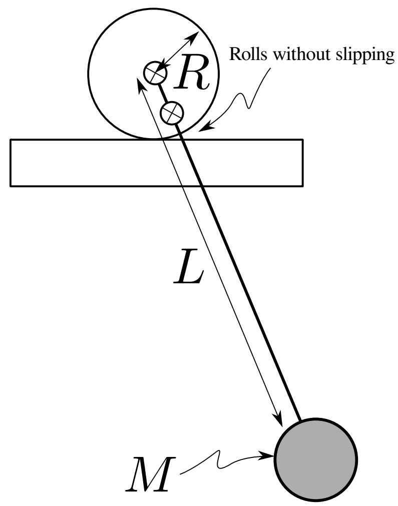

(a) Find the equation of motion for this system.

Solution:
We will begin by constructing a Lagrangian for this system. In doing so, we will use three coordinates: x and y will be the coordinates of the mass, expressed in a reference frame in which the center of the wheel is $(0, 0)$ when $\theta = 0$, where θ (the third coordinate) is the angle of the pendulum from the vertical. Obviously, only one of these three coordinates is independent. To begin with y:

$$y = -L\cos\theta \tag{36}$$

For x, we need to account first for the location of the wheel center, which is at $-R\theta$, and secondly for the location of the mass relative to the wheel center, which gives:

$$x = -R\theta + L\sin\theta \tag{37}$$

From these, we can find the potential energy

$$V = Mgy \tag{38}$$
$$= -MgL\cos\theta \tag{39}$$

and the kinetic energy

$$
\begin{aligned}
T &= \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) && (40)\\
&= \frac{1}{2}M\big((-R\dot{\theta} + L\cos\theta\dot{\theta})^2 + L^2\sin^2\theta\dot{\theta}^2\big) && (41)\\
&= \frac{1}{2}M\big(R^2 + L^2 - 2RL\cos\theta\big)\dot{\theta}^2 && (42)
\end{aligned}
$$

Finally, we put these together to obtain our Lagrangian:

$$
\begin{aligned}
\mathcal{L} &= T - V && (43)\\
&= \frac{1}{2}M\big(R^2 + L^2 - 2RL\cos\theta\big)\dot{\theta}^2 + MgL\cos\theta && (44)
\end{aligned}
$$

and we use the Euler-Lagrange equation:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{\partial\mathcal{L}}{\partial\theta} && (45)\\
\frac{d}{dt}\Big(M\big(R^2 + L^2 - 2RL\cos\theta\big)\dot{\theta}\Big) &= MRL\sin\theta\dot{\theta}^2 - MgL\sin\theta && (46)\\
\frac{d}{dt}\Big(\big(R^2 + L^2 - 2RL\cos\theta\big)\dot{\theta}\Big) &= \big(RL\dot{\theta}^2 - gL\big)\sin\theta && (47)\\
\big(R^2 + L^2 - 2RL\cos\theta\big)\ddot{\theta} + 2RL\sin\theta\dot{\theta}^2 &= \big(RL\dot{\theta}^2 - gL\big)\sin\theta && (48)\\
\big(R^2 + L^2 - 2RL\cos\theta\big)\ddot{\theta} &= -\big(RL\dot{\theta}^2 + gL\big)\sin\theta && (49)
\end{aligned}
$$

And there we have it, the equation of motion for our pendulum thing.

(b) Solve for the period of oscillation, as a function of the amplitude. Your solution may contain an integral.

Solution:
The easy way to solve this involves using a first integral of the motion (which is to say, energy conservation). The total energy is:

$$
\begin{aligned}
E &= T + V && (50)\\
&= \frac{1}{2}M\big(R^2 + L^2 - 2RL\cos\theta\big)\dot{\theta}^2 - MgL\cos\theta && (51)\\
&= \text{constant} && (52)
\end{aligned}
$$

Because this is constant, we can solve for $\dot{\theta}$ as a function of θ.

$$E = \frac{1}{2}M\big(R^2 + L^2 - 2RL\cos\theta\big)\dot{\theta}^2 - MgL\cos\theta \tag{53}$$

$$\dot{\theta}^2 = \frac{E + MgL\cos\theta}{\frac{1}{2}M\big(R^2 + L^2 - 2RL\cos\theta\big)} \tag{54}$$

$$\dot{\theta} = \sqrt{\frac{E + MgL\cos\theta}{\frac{1}{2}M\big(R^2 + L^2 - 2RL\cos\theta\big)}} \tag{55}$$

$$= \sqrt{2gL}\sqrt{\frac{\cos\theta - \cos\theta_0}{R^2 + L^2 - 2RL\cos\theta}} \tag{56}$$

$$= \sqrt{\frac{2gL}{R^2 + L^2}}\sqrt{\frac{\cos\theta - \cos\theta_0}{1 - \frac{2RL}{R^2+L^2}\cos\theta}} \tag{57}$$

where I have defined θ_0 as the maximum value of θ, which corresponds to the point where the energy is all potential. Now that we have $\dot{\theta}(\theta)$, we just need to integrate its inverse to find the period. It's easiest to just integrate over a quarter period:

$$\tau = 4\int_0^{\theta_0} \frac{dt}{d\theta} d\theta \tag{58}$$

$$= 4\int_0^{\theta_0} \frac{d\theta}{\dot{\theta}} \tag{59}$$

$$\tag{60}$$

$$= 4\sqrt{\frac{R^2 + L^2}{2gL}}\int_0^{\theta_0} \sqrt{\frac{1 - \frac{2RL}{R^2+L^2}\cos\theta}{\cos\theta - \cos\theta_0}} d\theta \tag{61}$$

And here is where we stop. You can see that the integral simplifies if $R = 0$, although not enough for you to be excited about solving it. If you also assume $|\theta_0| \ll 0$ then it simplifies further and you get the simple harmonic oscillator solution.