Chapter 9. Radiation

We discuss electromagnetic wave generation in this chapter. We consider the emission of radiation by localized systems of oscillating charge and current densities. Approximations are made for fields produced by slowly moving (nonrelativistic) charges. The results are widely applicable from the emission of radio waves from antennas and to the emission of light from atoms.

9.1 Fields and Radiation of a Localized Oscillating Source

We can define $\mathbf{B}$ in terms of a vector potential:

$$
\mathbf{B} = \nabla \times \mathbf{A}
$$

(9.1)

And define a scalar potential $\Phi$ satisfying

$$
\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}
$$

(9.2)

Imposing the Lorenz condition

$$
\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0
$$

(9.3)

we have the wave equations for the scalar and vector potentials:

$$
\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}
$$

(9.4)

and

$$
\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}
$$

(9.5)

Using the Green functions for the wave equations, we can express the retarded scalar potential as

$$
\Phi(x, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{[\rho(x', t')]_{\text{ret}}}{|x - x'|} d^3x'
$$

(9.6)

and the retarded vector potential as

$$
\mathbf{A}(x, t) = \frac{\mu_0}{4\pi} \int \frac{[\mathbf{J}(x', t')]_{\text{ret}}}{|x - x'|} d^3x'
$$

(9.7)

Where $[\ ]_{\text{ret}}$ means that $t' = t - |x - x'|/c$. Eqs. 9.6 and 9.7 indicate that at a given point $x$ and a given time $t$ the potentials are determined by the charge and current that existed at other points in space $x'$ at earlier times $t'$. The time appropriate to each source point is earlier than $t$ by $|x - x'|/c$, the time required to travel from source to field point $x$ with velocity $c$. In the above procedure it is essential to impose the Lorentz condition on the potentials; otherwise it would not be the simple wave equations.
We consider charge and current distributions which are harmonically periodic in their time dependence:

\[
\rho(x', t') = \rho(x') e^{-i \omega t'}
\]
\[
J(x', t') = J(x') e^{-i \omega t'}
\]  

(9.8)

The source is localized in the area of which dimension is of order \(d\) as shown in Fig 9.1.

![Fig 9.1 Arbitrarily moving charges localized in a volume of size \(\sim d\). The fields are to be calculated at \(x\).](image)

Substituting Eq. 9.8 into the solution for the vector potential \(A(x, t)\) in the Lorenz gauge

\[
A(x, t) = \frac{\mu_0}{4\pi} \int \frac{J(x', t - \frac{|x - x'|}{c})}{|x - x'|} d^3x' = A(x) e^{-i \omega t}
\]  

(9.9)

we have

\[
A(x) = \frac{\mu_0}{4\pi} \int \frac{J(x') e^{ik|x - x'|}}{|x - x'|} d^3x'
\]  

(9.10)

where \(k = \omega/c = 2\pi/\lambda\) is the wave number. The magnetic induction is given by

\[
B = \nabla \times A
\]

(9.11)

and making use of the Maxwell’s equation outside of the source

\[
\nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t}
\]  

(9.12)

we obtain

\[
E = \frac{ic}{k} \nabla \times B = \frac{ic}{k} \nabla \times [\nabla \times A]
\]  

(9.13)

We shall take the point of view that \(d\), i.e., \(r' = |x'|\), is infinitesimal compared with \(r = |x|\). Then, there are three spatial regions of interest:

Near (static) zone: \(d \ll r \ll \lambda\)

Intermediate (induction) zone: \(d \ll r \sim \lambda\)

Far (radiation) zone: \(d \ll \lambda \ll r\)

Near zone

In the near zone for \(kr \ll 1\), Eq. 9.10 reduces to

\[
A(x) \approx \frac{\mu_0}{4\pi} \int \frac{J(x')}{|x - x'|} d^3x'
\]  

(9.14)
This is the form of the magnetostatic vector potential. Therefore, the near fields are quasi-stationary, oscillating harmonically as $e^{-i\omega t}$, but otherwise static in character. Using the addition theorem, we can expand Eq. 9.14 as

$$A(x) \approx \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \int J(x') r' l Y_{lm}^*(\theta', \phi') \, d^3 x'$$

for the points outside the source ($r > r'$).

**Far zone**

In the far zone for $kr \gg 1$, we apply the condition $d \ll r$

$$|x - x'| \approx r - n \cdot x'$$

where $n$ is a unit vector in the direction of $x$. Then, Eq.9.10 reduces to

$$A(x) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int J(x') e^{-ikn \cdot x'} \, d^3 x'$$

This demonstrates that in the far zone the vector potential behaves as an outgoing spherical wave with an angular dependent coefficient. The fields are transverse to the radius vector and fall off as $1/r$. Since $d \ll \lambda$ (or $kd \ll 1$),

$$A(x) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int J(x') (n \cdot x')^n \, d^3 x'$$

and the magnitude of the $n$th term,

$$\frac{1}{n!} \int J(x')(kn \cdot x')^n \, d^3 x' \sim \frac{(kd)^n}{n!}$$

falls off rapidly with $n$.

**Electric monopole**

In the far zone Eq. 9.6 reduces to

$$\Phi(x, t) \approx \frac{1}{4\pi \epsilon_0} \frac{1}{r} \int \rho \left( x', t - \frac{|x - x'|}{c} \right) \, d^3 x' = \frac{1}{4\pi \epsilon_0} \frac{Q(t - r/c)}{r}$$

where $Q$ is the total charge of the source. Since charge is conserved, $Q$ must be independent of time. Thus electric monopole fields are intrinsically static. In other words, harmonic fields cannot include electric monopole terms.

**9.2 Electric Dipole Fields and Radiation**

As a first approximation we keep the first term in Eq. 9.18. We then find

$$A(x) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int J(x') \, d^3 x'$$
Note that here we apply the limit \(d \ll r\) independent of \(\lambda\), so that this approximation is valid both in the far and near zone. We evaluate the \(x\) component of \(\int J(x') \, d^3 x'\):

\[
\int j_x(x') \, d^3 x' = \int J(x') \cdot e_x' \, d^3 x' = \int \nabla' x' \, d^3 x' \\
= \int \nabla' \cdot [x'J(x')] \, d^3 x' - \int x' \nabla' \cdot J(x') \, d^3 x' \\
= \int [x'J(x')] \, da - \int x' \nabla' \cdot J(x') \, d^3 x' = -\int x' \nabla' \cdot J(x') \, d^3 x' \tag{9.22}
\]

Generalizing three dimensions, we obtain

\[
\int J(x') \, d^3 x' = -\int x' \nabla' \cdot J(x') \, d^3 x' \tag{9.23}
\]

Using the continuity equation, \(\nabla \cdot J = i\omega \rho\), we have

\[
A(x) = -\frac{i\omega \mu_0}{4\pi} \frac{e^{ikr}}{r} p \tag{9.24}
\]

where

\[
p = \int x' \rho(x') \, d^3 x' \tag{9.25}
\]

is the electric dipole moment, as defined in electrostatics. In order to determine \(B\), we take the curl of \(A\),

\[
B = -\frac{i\omega \mu_0}{4\pi} \nabla \left( \frac{e^{ikr}}{r} \right) \times p = \mu_0 c k^2 \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) n \times p \tag{9.26}
\]

where \(n = x/r\) is a radial unit vector. \(B\) is transverse to \(n\). From Eq. 9.13, we obtain

\[
E = \frac{ic}{k} \nabla \times B = \frac{ic}{k} \nabla \times \left[\mu_0 c k^2 \frac{e^{ikr}}{4\pi} \frac{1}{r} \left(1 - \frac{1}{ikr}\right) n \times p\right] \\
= \frac{1}{4\pi \varepsilon_0} \frac{e^{ikr}}{r} \left\{ k^2 (n \times p) \times n + [3n(n \cdot p) - p] \left(\frac{1}{r^2} - \frac{ik}{r}\right) \right\} \tag{9.27}
\]

The electric field has components parallel and perpendicular to \(n\).

**Far zone**

The fields take on the limiting forms,

\[
B = \frac{1}{4\pi \varepsilon_0} \frac{e^{ikr} k^2}{r} (n \times p) \tag{9.28}
\]

\[
E = \frac{1}{4\pi \varepsilon_0} \frac{e^{ikr}}{r} k^2 (n \times p) \times n = cB \times n
\]

showing the typical behavior or radiation fields. We can rewrite the electric field as
\[ E = \frac{1}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} k^2 [\mathbf{p} - (\mathbf{p} \cdot \mathbf{n})\mathbf{n}] = \frac{1}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} k^2 \mathbf{p}_p \] (9.29)

where \( \mathbf{p}_p \) is the component of the dipole moment perpendicular to \( \mathbf{n} \). Therefore,

\[ E(x, t) = \frac{1}{4\pi\varepsilon_0} \frac{k^2}{r} \mathbf{p}_p (t - \frac{r}{c}) \] (9.30)

\[ \mathbf{B}(x, t) = \frac{1}{c} \mathbf{n} \times E(x, t) \]

where \( \mathbf{p}_p(t - r/c) \) is \( \mathbf{p}_p \) evaluated at \( t - r/c \).

### Near zone

The fields are expressed as

\[ B = \frac{1}{4\pi\varepsilon_0} \frac{ik}{cr^2} (\mathbf{n} \times \mathbf{p}) \]

\[ E = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} [3\mathbf{n} (\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}] \]

The electric field, apart from its oscillations in time, is just the static electric dipole field. The magnetic induction \( c\mathbf{B} \approx kr\mathbf{E} \ll \mathbf{E} \), since \( kr \ll 1 \) in the near zone. Thus the fields in the near zone are dominantly electric in nature.

#### Dipole radiation power

In the far zone the time-averaged Poynting vector is

\[ \mathbf{S} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* = \frac{1}{2} c\varepsilon_0 |\mathbf{E}|^2 \mathbf{n} = \frac{c}{32\pi^2 \varepsilon_0} \frac{k^4}{r^2} |\mathbf{p}_p|^2 \mathbf{n} \] (9.32)

Then the time-averaged power radiated per unit solid angle by the oscillating dipole \( \mathbf{p} \) is

\[ \frac{dP}{d\Omega} = r^2 \mathbf{S} \cdot \mathbf{n} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2 \theta \] (9.33)

where \( \theta \) is the angle between \( \mathbf{p} \) and \( \mathbf{n} \), and \( Z_0 = \sqrt{\mu_0/\varepsilon_0} \approx 376.7 \Omega \) is the vacuum impedance. Integrating over a sphere, we find the average amount of energy radiated by the dipole per unit time:

\[ P = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \int_0^{2\pi} \int_0^\pi \sin^3 \theta d\theta d\phi = \frac{c^2 Z_0 k^4}{12\pi} |\mathbf{p}|^2 \] (9.34)

#### Dipole antenna

We calculate the radiation field due to the simple short dipole antenna shown in Fig. 9.2. We assume that the overall length of antenna is \( a \) and that \( a \ll \lambda \). The current in the antenna is taken to be

\[ I(z', t) = I_0 \left( 1 - \frac{2|z'|}{a} \right) e^{-i\omega t} \] (9.35)
In this model, the current is in the same direction in each half of the antenna and falls off linearly as we approach the ends.

\[
p = \frac{i}{\omega} \int \mathbf{J}(\mathbf{x}') \, d^3x' = \frac{iI_0}{\omega} \left[ \int_0^{a/2} \left( 1 - \frac{2z'}{a} \right) dz' + \int_{-a/2}^0 \left( 1 + \frac{2z'}{a} \right) dz' \right] \mathbf{e}_z = \frac{iI_0a}{2\omega} \mathbf{e}_z
\]

(9.36)

To obtain the dipole moment, we can make use of Eq. 9.21 and Eq. 9.24.

The angular distribution of radiated power is

\[
\frac{dP}{d\Omega} = \frac{Z_0}{128\pi^2} (ka)^2 I_0^2 \sin^2 \theta
\]

(9.37)

and the total power radiated is

\[
P = \frac{Z_0}{48\pi} (ka)^2 I_0^2
\]

(9.38)

Thus for a given maximum current the radiation power increases as the square of the frequency in the situation where \( a \ll \lambda \). A resistance \( R \) carrying a current \( I_0 e^{-i\omega t} \) dissipates energy at an average rate \( P = \frac{1}{2} RI_0^2 \). Comparing this with Eq. 9.38, we see that it is sensible to define the radiation resistance of a dipole by

\[
R_{rad} = \frac{Z_0}{24\pi} (ka)^2 \approx 5(ka)^2
\]

(9.39)

### 9.3 Magnetic Dipole and Electric Quadrupole Radiation

In the next order of approximation we consider the second term in the expansion in Eq. 9.18:

\[
\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \int \mathbf{J}(\mathbf{x}') (\mathbf{n} \cdot \mathbf{x}') \, d^3x'
\]

(9.40)
It is convenient to break up the integrand as follows:

\[
J(n \cdot x') = \frac{1}{2} [J(n \cdot x') - x'(J \cdot n)] + \frac{1}{2} [J(n \cdot x') + x'(J \cdot n)]
\]

\[
= \frac{1}{2} [n \times (J \times x')] + \frac{1}{2} [J(n \cdot x') + x'(J \cdot n)]
\]

The part of the vector potential corresponding to the first term is called the magnetic dipole potential, and the part corresponding to the second term is called the electric quadrupole potential.

**Magnetic dipole radiation**

The magnetic dipole potential is expressed as

\[
A(x) = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} n \times \int \frac{1}{2} [x' \times J(x')] d^3x' = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} n \times m
\]

Where \(m\) is the magnetic dipole moment,

\[
m = \int \frac{1}{2} [x' \times J(x')] d^3x'
\]

In the far zone we find the magnetic induction

\[
B = \nabla \times A \approx \frac{\mu_0}{4\pi} \frac{k^2 e^{ikr}}{r} (n \times m) \times n
\]

\[
= \frac{\mu_0}{4\pi} \frac{k^2 e^{ikr}}{r} [m - (m \cdot n)n] = \frac{\mu_0}{4\pi} \frac{k^2}{r} m_p (t - \frac{r}{c})
\]

where \(m_p\) is the component of the dipole moment perpendicular to \(n\) evaluated at \(t - r/c\).

Note the similarity to the electric dipole field in Eq. 9.29. The electric field takes the form

\[
E = \frac{ic}{k} \nabla \times B = cB \times n
\]

**Circular current loop**

The circular current loop shown in Fig. 9.3 is a magnetic dipole. The loop of circumference \(a\) has the current \(I(t) = I_0 e^{-i\omega t}\). We assume that \(a \ll \lambda\).
The magnetic moment of the loop is
\[ \mathbf{m} = I_0 \frac{a^2}{4\pi} \mathbf{e}_z \] (9.46)

Hence the angular distribution of radiated power (see Eq. 9.33) is
\[ \frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2} k^4 |\mathbf{m}|^2 \sin^2 \theta = \frac{Z_0}{512\pi^3} (ka)^4 I_0^2 \sin^2 \theta \] (9.47)

It is instructive to compare this result with Eq. 9.37. For comparable currents and a comparable size, the ratio of magnetic dipole radiation to electric dipole is
\[ \left( \frac{dP}{d\Omega} \right)_{MD} \left( \frac{dP}{d\Omega} \right)_{ED} = \left( \frac{ka}{\lambda} \right)^2 \] (9.48)

The intensity of magnetic dipole radiation is thus characteristically smaller by a factor of \((a/\lambda)^2\) than that of electric dipole radiation.

**Electric quadrupole radiation**

Referring to Eqs. 9.40 and 9.41, we can write the electric quadrupole potential as
\[ \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} e^{ikr} \int \left[ J \cdot (\mathbf{n} \cdot \mathbf{x}') + \mathbf{x}' (J \cdot \mathbf{n}) \right] d^3 x' \] (9.49)

We first examine the \(x\) component of the first part of the integral.
\[ \int J_\mathbf{x} (\mathbf{n} \cdot \mathbf{x}') \ d^3 x' = \int (\mathbf{n} \cdot \mathbf{x}') \mathbf{J} \cdot \mathbf{e}_x \ d^3 x' = \int (\mathbf{n} \cdot \mathbf{x}') \mathbf{J} \cdot \nabla' x' \ d^3 x' \]
\[ = \int \nabla' \cdot [x'(\mathbf{n} \cdot \mathbf{x}')] \mathbf{J} \ d^3 x' - \int x' \nabla' \cdot (\mathbf{n} \cdot \mathbf{x}') \mathbf{J} \ d^3 x' \]
\[ = \int [x'(\mathbf{n} \cdot \mathbf{x}')] \mathbf{J} \cdot \mathbf{e}_x \ d\mathbf{a} - \int x' \nabla' \cdot (\mathbf{n} \cdot \mathbf{x}') \mathbf{J} \ d^3 x' \]

We eliminate the first integral by noting that \(\mathbf{J} = 0\) on the surface. Hence
\[ \int J_\mathbf{x} (\mathbf{n} \cdot \mathbf{x}') \ d^3 x' = -\int x' \nabla' \cdot (\mathbf{n} \cdot \mathbf{x}') \mathbf{J} \ d^3 x' \]
\[ = -\int x' \nabla' (\mathbf{n} \cdot \mathbf{x}') \cdot \mathbf{J} \ d^3 x' - \int x'(\mathbf{n} \cdot \mathbf{x}') \nabla' \cdot \mathbf{J} \ d^3 x' \]

Since \(\nabla' (\mathbf{n} \cdot \mathbf{x}') = \mathbf{n}\) and \(\nabla' \cdot \mathbf{J}(\mathbf{x}') = -i\omega \rho(\mathbf{x}')\), we have
\[ \int J_\mathbf{x} (\mathbf{n} \cdot \mathbf{x}') \ d^3 x' = -\int x'(\mathbf{n} \cdot \mathbf{x}') d^3 x' - i\omega \int x'(\mathbf{n} \cdot \mathbf{x}') \rho(\mathbf{x}') \ d^3 x' \]

Generalizing to three dimensions, we obtain
\[ \int [J (\mathbf{n} \cdot \mathbf{x}') + x'(\mathbf{n} \cdot \mathbf{x})] d^3 x' = -i\omega \int x'(\mathbf{n} \cdot \mathbf{x}') \rho(\mathbf{x'}) \ d^3 x' \] (9.50)
Substituting back into Eq. 9.49, we find

$$A(x) = -\frac{\mu_0 c k^2}{8\pi r} e^{ikr} \int x'(n \cdot x') \rho(x') d^3 x'$$  \hspace{1cm} (9.51)$$

Taking the curl of $A$ and ignoring terms of order $1/r^2$, we find the magnetic induction in the radiation zone,

$$B = i k n \times A = -\frac{i \mu_0 c k^3}{24\pi r} e^{ikr} \int n \times x'(n \cdot x') \rho(x') d^3 x'$$ \hspace{1cm} (9.52)$$

We add to the integrand a term, $n \times n r^2 \rho(x')$, which is zero, yet helps us put the field in a more conventional form. Then we have

$$B = i k n \times A = -\frac{i \mu_0 c k^3}{24\pi r} n \times \int [3x'(n \cdot x') - n r^2] \rho(x') d^3 x'$$ \hspace{1cm} (9.53)$$

The integral on the right side of Eq. 9.53 can be written as a vector $Q(n)$ which is the product of the quadrupole moment tensor $\overline{Q}$ and the vector $n$:

$$Q(n) = \overline{Q} \cdot n = \int [3x'(n \cdot x') - n r^2] \rho(x') d^3 x'$$ \hspace{1cm} (9.54)$$

where the tensor elements are

$$Q_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) \rho(x) d^3 x$$ \hspace{1cm} (9.55)$$

And the vector $Q(n)$ is defined as having components,

$$Q_\alpha = \sum_\beta Q_{\alpha\beta} n_\beta$$ \hspace{1cm} (9.56)$$

The magnetic induction is thus

$$B = -\frac{i \mu_0 c k^3}{24\pi r} \frac{e^{ikr}}{r} n \times Q(n)$$ \hspace{1cm} (9.57)$$

and, as usual the electric field is

$$E = c B \times n$$ \hspace{1cm} (9.58)$$

Charge distribution of cylindrical symmetry

When the charge distribution is cylindrically symmetrical, the off-diagonal elements of $Q$ vanish ($Q_{\alpha\beta} = 0$ for $\alpha \neq \beta$) and the diagonal elements can be written as

$$Q_{33} = Q_0, \quad Q_{11} = Q_{22} = -\frac{1}{2} Q_0$$ \hspace{1cm} (9.59)$$

with the quadrupole moment

$$Q_0 = \int (3z^2 - r^2) \rho(x) d^3 x$$ \hspace{1cm} (9.60)$$
The angular distribution of radiated power is

\[
\frac{dP}{d\Omega} = r^2 \mathbf{S} \cdot \mathbf{n} = \frac{cr^2 |B|^2}{2 \mu_0} = \frac{c^2 Z_0 k^6}{1152 \pi^2} |\mathbf{n} \times \mathbf{Q(n)}|^2
\]  

(9.61)

Since \( \mathbf{Q(n)} = (-\frac{1}{2} Q_0 n_x, -\frac{1}{2} Q_0 n_y, Q_0 n_z) \),

\[
|\mathbf{n} \times \mathbf{Q(n)}|^2 = \frac{1}{4} Q_0^2 (\frac{1}{2} n_x^2 + \frac{1}{2} n_y^2 + n_z^2)^2 = \frac{9}{4} Q_0^2 \sin^2 \theta \cos^2 \theta
\]

Thus we have

\[
\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512 \pi^2} Q_0^2 \sin^2 \theta \cos^2 \theta
\]

(9.62)

The radiation pattern is shown in Fig. 9.3.

The total power radiated by this quadrupole is

\[
P = \frac{c^2 Z_0 k^6}{960 \pi} Q_0^2
\]

(9.63)

Quadrupole radiator

As a simple example of a quadrupole radiator we consider the assembly of point charges shown in Fig. 9.4. The charge \(-2q\) is stationary at the origin, and two positive charges harmonically oscillate, each with amplitude \(a\), about the origin. The two positive charges are always on opposite sides of the origin and exchange positions every half-cycle.
Assuming that the charges are at their maximum amplitude at \( t = 0 \), we calculate the quadrupole moment,

\[
Q_0 = \int (3z^2 - r^2)[-2q\delta(x) + q\delta(x - ae_z) + q\delta(x + ae_z)]\,d^3x = 4qa^2
\]  

The angular distribution of radiated power is

\[
\frac{dP}{d\Omega} = \frac{c^2Z_0q^2a^4k^6}{32\pi^2}\sin^2 \theta \cos^2 \theta
\]

and the total radiation power is

\[
P = \frac{c^2Z_0q^2a^4k^6}{60\pi}
\]

It is instructive to compare this result with the electric dipole radiation power (Eq. 9.34). For electric dipole moment \( p = qa \),

\[
P_{ED} = \frac{c^2Z_0q^2a^2k^4}{12\pi}
\]

Then the ratio of electric quadrupole radiation to electric dipole is

\[
\frac{P_{QD}}{P_{ED}} = \frac{(ka)^2}{5} \approx 8\left(\frac{a}{\lambda}\right)^2
\]

The intensity of electric quadrupole radiation is thus characteristically smaller by a factor of \( (a/\lambda)^2 \) than that of electric dipole radiation. This result is similar to Eq. 9.48, indicating that electric quadrupole and magnetic dipole radiation have the same basic strength.
9.4 Center-Fed Linear Dipole Antenna

The restriction to source dimension small compared with $\lambda$ can be removed for certain radiating systems. As an example of such a system we consider a thin, linear antenna of length $a$ which is excited across a small gap at its midpoint (Fig 9.5).

![Fig 9.5 Center-fed, linear antenna](image)

The current along the antenna can be taken as sinusoidal in time and space with wave number $k = \omega/c$ and is symmetric on the two arms of the antenna. The current vanishes at the ends of the antenna.

$$I(z) = I \sin \left( \frac{ka}{2} - k|z| \right) \quad \text{for} \quad |z| < \frac{a}{2}$$

(9.69)

Then the vector potential in the far zone (Eq. 9.17) takes the form

$$A(x) = e_z \frac{\mu_0}{4\pi} \frac{I e^{ikr}}{r} \int_{\frac{a}{2}}^{\frac{a}{2}} \sin \left( \frac{ka}{2} - k|z'| \right) e^{-ikz'} \cos \theta \, dz'$$

(9.70)

The integration is straightforward:

$$A(x) = e_z \frac{\mu_0}{4\pi} \frac{2I e^{ikr}}{kr} \left[ \cos \left( \frac{ka}{2} \cos \theta \right) - \cos \left( \frac{ka}{2} \right) \right]$$

(9.71)

Since the magnetic induction in the radiation zone is given by $B = ikn \times A$, its magnitude is $|B| = k \sin \theta \, |A_z|$. Thus the time averaged power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = r^2 S \cdot n = \frac{cr^2 |B|^2}{2} \frac{Z_0 I^2}{\mu_0} \left| \cos \left( \frac{ka}{2} \cos \theta \right) - \cos \left( \frac{ka}{2} \right) \right|^2 \sin \theta$$

(9.72)
For the special values $ka = \pi (a = \lambda/2$, half-wave antenna) and $ka = 2\pi (a = \lambda$, full-wave antenna), the angular distributions are

$$\frac{dP}{d\Omega} = \frac{Z_0 l^2}{8\pi^2} \left\{ \frac{\cos^2 \left( \frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta}, \quad a = \frac{\lambda}{2} \right\},$$

$$\frac{dP}{d\Omega} = \frac{Z_0 l^2}{8\pi^2} \left\{ \frac{4 \cos^4 \left( \frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta}, \quad a = \lambda \right\} \quad (9.73)$$

Fig 9.6 Normalized radiation pattern of a center-fed linear dipole antenna with sinusoidal current distribution on the z-axis. Three of the sides of the cube correspond to intersections with the coordinate planes.