Problem Statement

Two point-masses, one with charge $Q$ and the other with charge $4Q$ are separated by a distance $L$. A third charged point-mass is placed so that the net force on all three charges is zero. What are the magnitude and direction of the electric field half-way between the first and third charge?

Hint: You will need to find the position and charge of the third mass.

Quick Solution

The new charge must have a different sign than the first two, so let’s call its charge $-q$, and the only place it’ll be balanced is between them.

The net force on the $-q$ charge is zero. Since one of the original charges has 4 times the charge of the other, it must be twice as far away in order for the net forces to be equal in magnitude. Hence, the new charge must be a distance $L/3$ from the $Q$ charge (putting it $2L/3$ from the $4Q$ charge).

Consider the $4Q$ charge. The net force on it must also be zero. The $-q$ charge is $2/3$ as far from it as the $Q$ charge, so we need $q = (2/3)^2Q = 4Q/9$ in order to get equal and opposite forces. (You could also get this from considering the net force on the $Q$ charge).

We want the electric field half-way between the $Q$ and $-q$ charge. We know that it must point along the axis connecting the three charges. Let $\vec{E}_Q$ be the field due to the $Q$ charge. Then, using proportions, we see that the net electric field is

$$\vec{E} = \vec{E}_Q \left(1 + \frac{4}{9} - \frac{4}{25}\right) = \frac{289}{225} \vec{E}_Q,$$

where the terms in parentheses correspond to the $Q$ charge itself, the $-4Q/9$ charge the same distance away on the opposite side, and the $4Q$ charge five times farther away on the opposite side. $\vec{E}_Q$ is of course $\vec{E}_Q = \frac{kQ}{(L/6)^2} = \frac{36kQ}{L^2}$, so the final electric field is

$$\vec{E} = \frac{1156kQ}{25L^2} \hat{x} = 46.24 \frac{kQ}{L^2} \hat{x},$$

where I’ve included the decimal term for those of you who distrust fractions.
If the new charge has the same sign as the other two, there’s no hope of balance as there will always be two of them pushing in approximately the same direction on the third. Hence, it must have a different sign than the first two; let’s call its charge $-q$.

If the net force on each charge is to be zero, we can’t introduce any components orthogonal to the forces the first two charges exert on one another. This means we have to put the $-q$ charge on the axis connecting them. Moreover, we cannot put it on the far side of either, as this would result in the $-q$ charge pulling in the same direction that the far charge is pushing, preventing the forces from canceling. Hence, we conclude that the $-q$ charge must be placed somewhere between the two original charges. Let’s call that the $x$-axis, with the $Q$ charge at the origin and $\hat{x}$ pointing from the $Q$ charge toward the $4Q$ charge.

Then, calling the distance from the $Q$ charge to the $-q$ charge “$x_0$”, the statement that the net force on all three charges is zero means we want

\[
0 = \frac{kqQ}{x_0^2} - \frac{4kQ^2}{L^2} \quad \text{Net force on Q charge}
\]

\[
0 = \frac{4kQ^2}{L^2} - k\frac{4qQ}{(L-x_0)^2} \quad \text{Net force on 4Q charge}
\]

\[
0 = \frac{kqQ}{(L-x_0)^2} - \frac{kqQ}{x_0^2} \quad \text{Net force on } -q \text{ charge}
\]

Adding the first of these to the second gives

\[
\frac{kqQ}{x_0^2} = \frac{4kQ^2}{(L-x_0)^2} \Rightarrow 4x_0^2 = (L-x_0)^2 \Rightarrow x = L/3,
\]

where the last bit comes from $x_0 > 0$. So that’s the position of the $-q$ charge. We now need to find $q$.

Adding the second equation to the third gives

\[
\frac{4kQ^2}{L^2} = \frac{kqQ}{x_0^2} = \frac{kqQ}{(L/3)^2} = \frac{9kqQ}{L^2}.
\]

Hence, $4Q = 9q \Rightarrow q = 4Q/9$.

We now know the location and charge of all three charges, so we can find the electric field. The charges are all in a line, so the electric field at a point on that line must also point along it. Since the $Q$ charge is at the origin and the $-q$ charge is at $x = L/3$, we want the field at $x = L/6$. Plugging this in, we get

\[
\vec{E} = \frac{kQ}{(L/6)^2} \hat{x} - \frac{-kq}{(L/6)^2} \hat{x} - \frac{4kQ}{(5L/6)^2} \hat{x} = \frac{1156kQ}{25L^2} \hat{x} = 46.24 \frac{kQ}{L^2} \hat{x}.
\]