6.1 CIRCULAR MOTION; CENTRIPETAL FORCE

6.1 A 0.3-kg mass attached to a 1.5 m-long string is whirled around in a horizontal circle at a speed of 6 m/s. (a) What is the centripetal acceleration of the mass? (b) What is the tension in the string? (Neglect gravity.)

(a) \[ a = \frac{v^2}{R} = \frac{(6 \text{ m/s})^2}{1.5 \text{ m}} = 24 \text{ m/s}^2 \]

(b) The tension in the string exerts the centripetal force required to keep the mass in circular motion. This force is \[ T = ma = (0.3 \text{ kg})(24 \text{ m/s}^2) = 7.2 \text{ N}. \]

6.2 A small ball is fastened to a string 24 cm long and suspended from a fixed point \( P \) to make a conical pendulum, as shown in Fig. 6-1. The ball describes a horizontal circle about a center vertically under point \( P \), and the string makes an angle of 15° with the vertical. Find the speed of the ball.

\[ T \cos 15^\circ = mg \quad T \sin 15^\circ = \frac{mu^2}{r} \quad \text{and hence} \quad \tan 15^\circ = \frac{v^2}{rg} \]

Since \( r = 24 \sin 15^\circ = 24(0.259) = 6.22 \text{ cm} \),

\[ \tan 15^\circ = \frac{v^2}{6.22(980)} \quad v = 40.4 \text{ cm/s} \]

6.3 In the Bohr model of the hydrogen atom an electron is pictured rotating in a circle (with a radius of \( 0.5 \times 10^{-10} \text{ m} \)) about the positive nucleus of the atom. The centripetal force is furnished by the electric attraction of the positive nucleus for the negative electron. How large is this force if the electron is moving with a speed of \( 2.3 \times 10^6 \text{ m/s} \)? (The mass of an electron is \( 9 \times 10^{-31} \text{ kg} \).)

Force = \( (9 \times 10^{-31} \text{ kg})(2.3 \times 10^6 \text{ m/s})^2/(5.0 \times 10^{-11} \text{ m}) = 9.5 \times 10^{-8} \text{ N} \)

6.4 Find the maximum speed with which an automobile can round a curve of 80-m radius without slipping if the road is unbanked and the coefficient of friction between the road and the tires is 0.81.

First draw a diagram showing the forces (Fig. 6-2). If \( mg \) is the weight of the automobile, then the normal force is \( N = mg \). The frictional force supplies the centripetal force \( F_c \).

\[ F_c = \mu \cdot N = 0.81 \cdot mg \]

Also,

\[ F_c = \frac{mu^2}{r} \quad 0.81 \cdot mg = \frac{mu^2}{80} \quad v^2 = 0.81 \times 80 \times 9.8 = 25.2 \text{ m/s} \]
6.5 What is the maximum velocity, in miles per hour, for an automobile rounding a level curve of 200-ft radius if \( \mu \), between tires and roadbed is 1.0.

\[
\mu, mg = \frac{mv^2}{r} \quad \mu = \frac{v^2}{rg} \quad v = \sqrt{\mu sr} = \sqrt{1.0 \times 32.2 \text{ ft/s}^2 \times 200 \text{ ft}} = 80 \text{ ft/s} \times \frac{30 \text{ mi/h}}{44 \text{ ft/s}} = 54.5 \text{ mi/h}
\]

6.6 A car is traveling 25 m/s (56 mi/h) around a level curve of radius 120 m. What is the minimum value of the coefficient of static friction between the tires and the road required to prevent the car from skidding?

\[
F = m \frac{v^2}{r} = F_i \leq \mu, mg \quad \mu_i \geq \frac{v^2}{gr} = \frac{(25 \text{ m/s})^2}{(9.8 \text{ m/s}^2)(120 \text{ m})} = 0.53
\]

6.7 Traffic is expected to move around a curve of radius 200 m at 90 km/h. What should be the value of the banking angle if no dependence is to be placed on friction?

![Fig. 6-3]

\[ w = mg = N \cos \theta \quad \text{and} \quad F_i = \frac{mv^2}{r} = N \sin \theta \]

Dividing the second by the first equation, \( \tan \theta = \frac{v^2}{rg} \). Substitute the data, changing kilometers per hour into meters per second:

\[
\tan \theta = \frac{(90 \text{ km/h} \times 1000 \text{ m/km}) \times 3600 \text{ s/h}}{200 \text{ m} \times 9.8 \text{ m/s}^2} = 0.319 \quad \text{so} \quad \theta = 17.7^\circ
\]

6.8 As indicated in Fig. 6-4, a plane flying at constant speed is banked at angle \( \theta \) in order to fly in a horizontal circle of radius \( r \). The aerodynamic lift force acts generally upward at right angles to the plane's wings and fuselage. This lift force corresponds to the tension provided by the string in a conical pendulum, or the normal force of a banked road. (a) Obtain the equation for the required banking angle \( \theta \) in terms of \( v \), \( r \), and \( g \). (b) What is the required angle for \( v = 60 \text{ m/s} \) (216 km/h) and \( r = 1.0 \text{ km} \)?

![Fig. 6-4]

(a) As in Probs. 6.2 and 6.7, \( \tan \theta = \frac{v^2}{rg} \).

\[ \theta = \tan^{-1} \left( \frac{60^2}{(1.0 \times 10^6)(9.8)} \right) = 20.2^\circ \]
A car goes around a curve of radius 48 m. If the road is banked at an angle of 15° with the horizontal, at what maximum speed in kilometers per hour may the car travel if there is to be no tendency to skid even on very slippery pavement?

\[
\tan \theta = \frac{v^2}{rg} \quad \tan 15^\circ = \frac{v^2}{48(9.8)}
\]

\[u = 0.268(48)(9.8) = 11.2 \, \text{m/s} - (11.2 \, \text{m/s})(0.001 \, \text{km/m})(3600 \, \text{s/h}) = 40.3 \, \text{km/h}
\]

A certain car of mass \(m\) has a maximum frictional force of 0.7 \(mg\) between it and pavement as it rounds a curve on a flat road \((\mu = 0.7)\). How fast can the car be moving if it is to successfully negotiate a curve of 15-m radius?

The centripetal force \((mv^2)/r\) must be supplied by the frictional force. In the limiting case, \(mv^2/r = f\) with \(f = 0.7 \, mg\). Thus, \(v^2 = 0.7rg\) and \(v = 10 \, \text{m/s}\).

A crate sits on the floor of a boxcar. The coefficient of friction between the crate and the floor is 0.6. What is the maximum speed that the boxcar can go around a curve of radius 200 m without causing the crate to slide?

As in other "unbanked-curve" problems (e.g., Prob. 6.5),

\[v_{\text{max}}^2 = \mu, gr = (0.6)(9.8 \, \text{m/s}^2)(200 \, \text{m}) = 1176 \, \text{m}^2/\text{s}^2 \quad v_{\text{max}} = \sqrt{1176 \, \text{m}^2/\text{s}^2} = 34.3 \, \text{m/s}
\]

A boy on a bicycle pedals around a circle of 22-m radius at a speed of 10 m/s. The combined mass of the boy and the bicycle is 80 kg. (a) What is the centripetal force exerted by the pavement on the bicycle? (b) What is the upward force exerted by the pavement on the bicycle? See Fig. 6-5.

\[F_c = \frac{mv^2}{r} = \frac{80(10)^2}{22} = 364 \, \text{N} \quad (b) \quad N = mg = 80(9.8) = 784 \, \text{N}
\]

Refer to Prob. 6.12. What is the angle that the bicycle makes with the vertical?

For the bicycle not to fall, the torque about the center of gravity must be zero—see Chaps. 9 and 10—which means that the vector force exerted by the ground must have a line of action passing through the center of gravity. Thus

\[\tan \theta = \frac{F_c}{N} = \frac{364}{784} = 0.4643 \quad \theta = 25^\circ
\]

A fly of mass 0.2 g sits 12 cm from the center of a phonograph record revolving at 33\(\frac{1}{2}\) rpm. (a) What is the
magnitude of the centripetal force on the fly? (b) What is the minimum value of the coefficient of static friction between the fly and the record required to prevent the fly from sliding off?

\[ v = \frac{2\pi r}{T} = 2\pi r \left( \frac{33.3 \text{ min}}{60 \text{ s/min}} \right) (12 \times 10^{-2} \text{ m}) = 0.419 \text{ m/s} \]

\[ F = ma = m \left( \frac{0.2 \times 10^{-3} \text{ kg}(0.419 \text{ m/s})^2}{0.12 \text{ m}} \right) = 2.92 \times 10^{-4} \text{ N} \]

\( F_s = 2.92 \times 10^{-4} \text{ N} \leq \mu mg \quad \mu_s = \frac{2.92 \times 10^{-4} \text{ N}}{(0.2 \times 10^{-3} \text{ kg})(9.8 \text{ m/s}^2)} = 0.149 \)

6.15 Find (a) the speed and (b) the period of a spaceship orbiting around the moon. The moon's radius is \(1.74 \times 10^6 \text{ m}\), and the acceleration due to gravity on the moon is \(1.63 \text{ m/s}^2\). (Assume that the spaceship is orbiting just above the moon's surface.)

\[ \frac{v^2}{R_m} = G \frac{M_m}{R_m^2} = g_m \quad v = \sqrt{g_m R_m} = \sqrt{(1.63 \text{ m/s}^2)(1.74 \times 10^6 \text{ m})} = 1.68 \times 10^4 \text{ m/s} = 1.68 \text{ km/s} \]

(b) The circumference of the orbit is

\[ d = 2\pi R_m = (6.28)(1.74 \times 10^6 \text{ m}) = 1.09 \times 10^7 \text{ km} \]

so the period is

\[ \frac{t}{d} = \frac{1.09 \times 10^7 \text{ km}}{1.68 \text{ km/s}} = 6.5 \times 10^4 \text{ s} = 108 \text{ min} \]

6.16 At the equator, the effective value of \(g\) is smaller than at the poles. One reason for this is the centripetal acceleration due to the earth's rotation. The magnitude of the centripetal acceleration must be subtracted from the magnitude of the acceleration due purely to gravity in order to obtain the effective value of \(g\).

(a) Calculate the fractional diminution of \(g\) at the equator as a result of the earth's rotation. Express your result as a percentage. (b) How short would the earth's period of rotation have to be in order for objects at the equator to be "weightless" (that is, in order for the effective value of \(g\) to be zero)? (c) How would the period found in part (b) compare with that of a satellite skimming the surface of an airless earth?

\[ a = \frac{v^2}{R_e} = 4\pi^2 R_e / T^2 = 3.37 \times 10^{-2} \text{ m/s}^2 \]

Therefore \(a/g = 3.44 \times 10^{-3}\). Since \(g_{\text{eq}} = g - a\), the fractional diminution is \((g - g_{\text{eq}})/g = a/g = 0.344\) percent.

(b) In order that \(g_{\text{eq}} = 0\), we need \(a = g = 4\pi^2 R_e / T^2\). Solving for \(T\), we find \(T = 2\pi \sqrt{R_e/g} = 5.06 \times 10^3 \text{ s} = 84.4 \text{ min}\). (c) Since an orbiting satellite has \(ma = mg\) and \(g_{\text{eq}} = 0\), its period equals \(T_e\).

6.17 A particle is to slide along the horizontal circular path on the inside of the funnel shown in Fig. 6-6. The surface of the funnel is frictionless. How fast must the particle be moving (in terms of \(r\) and \(\theta\)) if it is to execute this motion?
6.18 An automobile moves around a curve of radius 300 m at a constant speed of 60 m/s [Fig. 6-7(a)]. (a) Calculate the resultant change in velocity (magnitude and direction) when the car goes around the arc of 60°. (b) Compare the magnitude of the instantaneous acceleration of the car to the magnitude of the average acceleration over the 60° arc.

(a) From Fig. 6-7(b), \( \Delta v = 60 \text{ ms} \) and \( \Delta \theta \) makes a 120° angle with \( v_A \). (b) The instantaneous acceleration has magnitude

\[
a = \frac{v^2}{r} = \frac{60^2}{300} = 12 \text{ m/s}^2
\]

The time average acceleration is \( \bar{a} = \Delta v / \Delta t \). Since

\[
\Delta t = \frac{\Delta s}{v} = \frac{\Delta \theta}{\theta} = \frac{300(\pi/3)}{60} = \frac{5\pi}{3} \text{ s}
\]

we have

\[
\bar{a} = \frac{\Delta v}{\Delta t} = \frac{60}{5\pi/3} = 11.5 \text{ m/s}^2
\]

Fig. 6-7

6.19 While driving around a curve of 200 m radius, an engineer notes that a pendulum in the car hangs at an angle of 15° to the vertical. What should the speedometer read (in kilometers per hour)?

\[
T \sin \theta = \frac{m v^2}{r} \quad T \cos \theta = mg \quad \text{where} \quad T = \text{tension}. \quad \text{Thus} \quad \tan \theta = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{rg \tan \theta} = 23 \text{ m/s} = 82.5 \text{ km/h}
\]

6.20 The bug shown in Fig. 6-8(a) has just lost its footing near the top of the stationary bowling ball. It slides down the ball without appreciable friction. Show that it will leave the surface of the ball at the angle \( \theta = \arccos \frac{1}{2} \approx 48° \).

Fig. 6-8
1. The centripetal force is given by

\[ \frac{mv^2}{r} = mg \cos \theta \quad F_c \]

At angle \( \theta \), the decrease in potential energy, \( mgh = mgr(1 - \cos \theta) \), must equal the increase in kinetic energy, \( \frac{mv^2}{2} \); hence,

\[ \frac{mv^2}{r} = 2mg(1 - \cos \theta) \]

(2)

Together, (1) and (2) give

\[ 3mg \cos \theta - F_c = 2mg \]

(3)

At the instant the bug loses contact with the ball, \( F_c = 0 \) and (3) yields \( \cos \theta = \frac{1}{3} \).

6.21 A 180-lb pilot is executing a vertical loop of radius 2000 ft at 350 mi/h. With what force does the seat press upward against him at the bottom of the loop?

\[ F - mg = \frac{mv^2}{r} \quad \text{or} \quad F = \frac{mv^2}{r} + mg \]

First we change miles per hour to feet per second:

\[ 350 \text{ mi/h} = 513 \text{ ft/s} \]

Substitute values:

\[ F = \frac{180 \text{ lb} \times (513 \text{ ft/s})^2}{32.2 \text{ ft/s}^2 \times 2000 \text{ ft}} + 180 \text{ lb} = 915 \text{ lb} \]

6.22 How many g's must the pilot of the preceding problem withstand at the bottom of the loop?

\[ a_c = \frac{(513 \text{ ft/s})^2}{2000 \text{ ft}} = 132 \text{ ft/s}^2 \]

Dividing this result by \( g \) (32.2 ft/s²), we obtain

\[ \frac{132 \text{ ft/s}^2}{32.2 \text{ ft/s}^2} = 4.1 \text{ g} \]

6.23 The designer of a roller coaster wishes the riders to experience "weightlessness" as they round the top of one hill. How fast must the car be going if the radius of curvature at the hilltop is 20 m?

\[ \text{To experience weightlessness, the gravitational force } mg \text{ must exactly equal the required centripetal force } \frac{mv^2}{r}. \text{ Equating the two and solving for } v \text{ gives } 14 \text{ m/s}. \]

6.24 A huge pendulum consists of a 200-kg ball at the end of a cable 15 m long. If the pendulum is drawn back to an angle of 37° and released, what maximum force must the cable withstand as the pendulum swings back and forth?

\[ \text{The maximum tension will occur at the bottom when the cable must furnish a force } mg + \frac{mv^2}{r}. \text{ To reach the bottom, the mass falls a distance } h = (15 - 15 \cos 37°) = 3.0 \text{ m}. \text{ Its speed there will be } v = (2gh)^{1/2} = (6g)^{1/2}. \text{ Therefore the tension will be } T = 200g + 200(6g)/15 = 2740 \text{ N}. \]

5.2 LAW OF UNIVERSAL GRAVITATION; SATELLITE MOTION

5.25 Two 16-lb shot spheres (as used in track meets) are held 2 ft apart. What is the force of attraction between them?

\[ \text{In American engineering units, } 16 \text{ lb} \Rightarrow 0.497 \text{ slug and Newton's law of gravitation has the form } F = G\left(\frac{m_1m_2}{d^2}\right), \text{ with } G = 3.44 \times 10^{-8} \text{ lb} \cdot \text{ft}^2/\text{slug}^2. \text{ Thus,} \]

\[ F = \left( 3.44 \times 10^{-8} \text{ lb} \cdot \text{ft}^2/\text{slug}^2 \right) \left( \frac{0.497 \text{ slug} \times 0.497 \text{ slug}}{(2 \text{ ft})^2} \right) = 2.12 \times 10^{-9} \text{ lb} \]
CHAPTER 6

6.26 Calculate the force of attraction between two 90-kg spheres of metal spaced so that their centers are 40 cm apart.

In SI units the gravitational constant has the value \( G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \).

\[
F = G \frac{m_1 m_2}{r^2} = (6.67 \times 10^{-11})(90)(90) = 3.38 \times 10^{-6} \text{ N}
\]

6.27 Compute the mass of the earth, assuming it to be a sphere of radius 6370 km.

Let \( M \) be the mass of the earth, and \( m \) the mass of a certain object on the earth’s surface. The weight of the object is equal to \( mg \). It is also equal to the gravitational force \( G(Mm)/r^2 \), where \( r \) is the earth’s radius. Hence, \( mg = G[Mm/r^2] \), from which

\[
M = \frac{gr^2}{G} = \frac{(9.8 \text{ m/s}^2)(6.37 \times 10^8 \text{ m})^2}{6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2} = 6.0 \times 10^{24} \text{ kg}
\]

6.28 The average density of solids near the surface of the earth is \( \rho \approx 4 \times 10^3 \text{ kg/m}^3 \). On the (crude) assumption of a spherical planet of uniform density \( \rho \), calculate the gravitational constant \( G \).

The mass of the spherical earth is given by \( m = \frac{4}{3} \pi \rho R_e^3 \). Insert this value into \( g = Gm/r^2 \) and solve for \( G \), obtaining:

\[
G = \frac{3g}{4\pi \rho R_e} = \frac{3 \times 9.8 \text{ m/s}^2}{4 \pi \times 4 \times 10^3 \text{ kg/m}^3 \times 6.4 \times 10^8 \text{ m}} = 9 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2
\]

This calculation most certainly overestimates \( G \), since \( \rho \) (and hence \( m \)) are underestimates.

6.29 A mass \( m_1 = 1 \text{ kg} \) weighs one-sixth as much on the surface of the moon as on the earth. Calculate the mass \( m_2 \) of the moon. The radius of the moon is \( 1.738 \times 10^6 \text{ m} \).

On the moon, \( m_1 \) weighs \( \frac{1}{6}(9.8 \text{ N}) \).

\[
w_1 = G \frac{m_1 m_2}{r^2} = \frac{1}{6} (9.8) = 6.67 \times 10^{-11} \times \frac{1 \times m_2}{(1.738 \times 10^6)^2}
\]

\[
m_2 = \frac{(9.8)(1.738 \times 10^6)^2}{(6)(6.67 \times 10^{-11})} = 7.4 \times 10^{22} \text{ kg}
\]

6.30 The earth’s radius is about 6370 km. An object that has a mass of 20 kg is taken to a height of 160 km above the earth’s surface. (a) What is the object’s mass at this height? (b) How much does the object weigh (i.e., how large a gravitational force does it experience) at this height?

(a) The mass is the same as that on the earth’s surface. (b) As long as we are outside the earth’s surface, the weight (force of gravity) varies inversely as the square of the distance from the center of the earth. Indeed \( w = GmM/r^2 \), where \( m \), \( M \) are the masses of object and earth, respectively, and \( r \) is the distance to the center of the earth. Thus \( w_2/w_1 = (r_1/r_2)^2 \), since \( G \), \( m \), \( M \) are constant in this problem. For our case we set \( r_1 = 6370 \text{ km} \) and \( r_2 = 6530 \text{ km} \) and \( w_2 = (20 \text{ kg})(9.8 \text{ m/s}^2) = 196 \text{ N} \). This gives \( w_2 = 186.5 \text{ N} \). Note that we could use the fact that \( w = mg \) and that \( m \) is constant to find the acceleration of gravity at the two heights. That is, \( g_2/g_1 = r_1^2/r_2^2 \).

6.31 The radius of the earth is about 6370 km, while that of Mars is about 3440 km. If an object weighs 200 N on earth, what would it weigh, and what would be the acceleration due to gravity on Mars? Mars has a mass 0.11 that of earth.

Newton’s law of gravitation, \( w = GmM/r^2 \), gives \( w_2/w_1 = (M_2/M_1)(r_1^2/r_2^2) \). Letting 1 refer to earth and 2 refer to Mars, we have \( w_2 = 0.11(6370/3440)^2(200 \text{ N}) = 75 \text{ N} \). The acceleration is gotten from \( w_2/w_1 = g_2/g_1 \), or \( g_2 = (75/200)(9.8 \text{ N}) = 3.7 \text{ m/s}^2 \).

6.32 The moon orbits the earth in an approximately circular path of radius \( 3.8 \times 10^8 \text{ m} \). It takes about 27 days to complete one orbit. What is the mass of the earth as obtained from these data?

The gravitational attraction between the earth and moon provides the centripetal force; therefore, \( mv^2/r = GMm/r^2 \), where \( M \) is the earth’s mass. Then \( M = v^2 r/G = \omega^2 r^3/G \). Now \( \omega = 1 \text{ rev/27 days} = 2.7 \times 10^{-8} \text{ rad/s} \), \( r = 3.8 \times 10^8 \text{ m} \), and \( G = 6.7 \times 10^{-11} \) in SI. Solving for \( M \), it is \( 5.0 \times 10^{24} \text{ kg} \). (Compare Prob. 6.27.)
6.33 The sun's mass is about $3.2 \times 10^8$ times the earth's mass. The sun is about 400 times as far from the earth as the moon is. What is the ratio of the magnitude of the pull of the sun on the moon to that of the pull of the earth on the moon? It may be assumed that the sun--moon distance is constant and equal to the sun--earth distance.

Let $m$ denote the moon's mass, $M_s$ the sun's mass, $M_e$ the earth's mass, $r_{sm}$ the center-to-center distance from the sun to the moon, and $r_{em}$ the center-to-center distance from the earth to the moon. We let $F_{sm}$ denote the magnitude of the gravitational force exerted on the moon by the sun, and $F_{em}$ denote the magnitude of the gravitational force exerted on the moon by the earth. Then $F_{sm} = GM_s m / r_{sm}^2$ and $F_{em} = GM_e m / r_{em}^2$, so that

$$\frac{F_{sm}}{F_{em}} = \frac{M_s r_{em}^2}{M_e r_{sm}^2}$$

Using the given numerical values, we find $F_{sm} / F_{em} = 2$.

6.34 Estimate the size of a rocky sphere with a density of 3.0 g/cm$^3$ from the surface of which you could just barely throw away a golf ball and have it never return. (Assume your best throw is 40 m/s.)

The escape speed $v_0$ from a sphere of radius $R$ and mass $M$ is given by the energy-conservation equation

$$\frac{1}{2} m v_0^2 = \frac{G m M}{R}$$

Substitution of $M = \rho \frac{4}{3} \pi R^3$ and solution for $R$ gives

$$R = v_0 \sqrt{\frac{3}{8\pi G \rho}}$$

If $\rho = 3 \times 10^6$ kg/m$^3$, then $R = 0.77 \times 10^6$ $v_0$, where $R$ is in meters and $v_0$ is in meters per second. Estimating the highest speed at which a human can throw a golf ball as about 40 m/s, we find $R = 3 \times 10^6$ m = 30 km.

Newton, without knowledge of the numerical value of the gravitational constant $G$, was nevertheless able to calculate the ratio of the mass of the sun to the mass of any planet, provided the planet has a moon.

(a) Show that for a circular orbit

$$\frac{M_s}{M_p} = \left( \frac{R_p}{R_m} \right)^3 \left( \frac{T_p}{T_m} \right)^2$$

where $M_s$ is the mass of the sun, $M_p$ the mass of the planet, $R_p$ the distance of the planet from the sun, $R_m$ the distance of the moon from the planet, $T_p$ the period of the moon around the planet, and $T_p$ the period of the planet around the sun. (b) If the planet is the earth, $R_p = 1.50 \times 10^8$ km, $R_m = 3.85 \times 10^8$ km, $T_m = 27.3$ days, and $T_p = 365.2$ days. Calculate $M_s / M_p$.

(a) Applying Newton's second law and the law of gravitation to each orbit, we find (expressing centripetal force in terms of period, $T$, using $v = 2\pi R / T$),

$$\frac{4\pi^2 M_p R_p}{T_p^2} = \frac{G M_s m}{R_p^2} \quad \text{and} \quad \frac{4\pi^2 M_m R_m}{T_m^2} = \frac{G M_p m}{R_m^2}$$

where $m$ is the mass of the satellite. Solving the above equations for $M_s / M_p$, we obtain

$$\frac{M_s}{M_p} = \left( \frac{4\pi^2 R_p^3}{G T_p^2} \right) \left( \frac{R_p}{R_m} \right)^3 \left( \frac{T_m}{T_p} \right)^2$$

as desired. (b) Inserting the given numerical values, we find

$$\frac{M_s}{M_p} = \left( \frac{1.50 \times 10^8}{3.85 \times 10^8} \right)^3 \left( \frac{27.3}{365.2} \right)^2 = 3.30 \times 10^9$$

6.36 (a) Find the orbital period of a satellite in a circular orbit of radius $r$ about a spherical planet of mass $M$. (b) For a low-altitude orbit ($r \geq r_p$), show that for a given average planetary density $\langle \rho \rangle$ the orbital period is independent of the size of the planet.
102 CHAPTER 6

I (a) Applying Newton's second law to the circular orbit, we have

\[ \frac{mv^2}{r} = \frac{4\pi^2m}{T^2} = \frac{GMm}{r^2} \]

where \( m \) is the satellite mass, \( v \) is its orbital speed, and \( T \) is its orbital period. Solving the above equation for the period, we obtain

\[ T = \frac{2\pi r^3}{\sqrt{GM}} \]

or \( T^2 \propto r^3 \), which is Kepler's third law. (b) Since \( M = 4\pi (\rho) r_e^3/3 \), the above equation for the period yields

\[ T = \frac{2\pi r_e^2}{\sqrt{(4\pi G(\rho) r_e^3)/3}} = \sqrt{\frac{3\pi}{G(\rho)}} \]

which shows that the period of a low-altitude satellite is determined solely by the average density of the planet.

6.37 The rings of Saturn consist of myriad small particles, with each particle following its own circular orbit in Saturn's equatorial plane. The inner edge of the innermost ring is about 70000 km from Saturn's center; the outer edge of the outermost ring is about 135000 km from the center. Find the orbital period of the outermost particles as a multiple of the orbital period of the innermost particles.

I We denote the innermost and outermost orbital radii and periods by \( R_i \), \( R_o \), \( T_i \), and \( T_o \), respectively. Applying Kepler's third law [see Prob. 6.36(a)] to the ring system, we obtain

\[ \frac{T_o}{T_i} = \left( \frac{R_o}{R_i} \right)^{5/2} = \left( \frac{135}{70} \right)^{5/2} = 2.68 \]

That is, \( T_o = 2.68 T_i \).

6.38 Refer to Prob. 6.37. Spectroscopic studies indicate that the outermost particles have a speed of 17 km/s. Find the mass of Saturn. Express your result in kilograms and as a multiple of the earth's mass.

I Applying Newton's second law and the law of gravitation to a particle of mass \( m \) at the outer edge of the rings, we have

\[ \frac{mv^2}{r} = \frac{GMm}{r^2} \]

where \( M \) is Saturn's mass. Therefore

\[ M_s = \frac{v^2 R_o}{G} = \frac{(17 \times 10^3 \text{ m/s})^2 (1.35 \times 10^9 \text{ m})}{6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}} = 5.85 \times 10^{26} \text{ kg} = 97.7 M_e \]

6.39 The acceleration due to gravity on the moon is only one-sixth that on earth. If the earth and moon are assumed to have the same average composition, what would you predict the moon's radius to be in terms of the earth's radius \( R_e \)?

I Since \( g = GM/R^2 \),

\[ \frac{1}{6} g_e = \frac{R_e^2 M_e}{R_m^2 M_m} = \frac{R_e^2 R_o^3}{R_m^2 R_e^3} = \frac{R_m}{R_e} \]

or \( R_m = \frac{1}{6} R_e \). (Actually \( \rho \) of the moon is about three-fifths that of earth, so that \( R_m = 0.27 R_e \).

6.40 Two identical coins of mass 8 g are 50 cm apart on a tabletop. How many times larger is the weight of one coin than the gravitational attraction of the other coin for it?

I The force between the coins is \( Gm^2/d^2 \); when dividing the weight \( mg \) by this value, the ratio

\[ = \frac{gd^2}{Gm} = 9.8(0.500)^2/(6.67 \times 10^{-11})(0.008) = 4.6 \times 10^{12} \]

6.41 There is a point along the line joining the center of the earth to the center of the moon at which the two gravitational forces cancel. Find this point's distance, \( x \), from the earth's center. Use \( D \) for the earth–moon
distance, and \( m_e \) and \( m_m \) as the masses of earth and moon, respectively.

\[
I \quad \frac{Gm_e}{x^2} = \frac{Gm_m}{(D-x)^2} \quad x(m_e - m_m) - \frac{2Dm_e x + m_e D^2}{m_e - m_m} = 0 \quad x = \frac{D(m_e - (m_m m_e)^{1/2})}{m_e - m_m}
\]

6.42 Communication satellites are placed in orbit above the equator in such a way that they remain stationary above a given point on earth below. How high above the surface of the earth is such a synchronous orbit? (\( R_e = 6400 \text{ km}, \; M_e = 5.98 \times 10^{24} \text{ kg} \).)

I The satellite must have the same angular velocity, \( \omega = 1 \text{ rev/day} = 7.27 \times 10^{-5} \text{ rad/s} \), about the earth's center as has the earth itself. As the gravitational force is the centripetal force that keeps the satellite in orbit, \( GMM/(R_e + h)^2 = m_e \omega^2 (R_e + h) \). First solve for \( (R_e + h) \); then find \( h = 35800 \text{ km} \), or about \( 5.6R_e \).

6.43 Three identical point masses \( M \) lie in the xy plane at points \((0.0), (0.0, 0.20 \text{ m}), \) and \((0.20 \text{ m}, 0) \). Find the components of the gravitational force on the mass at the origin.

\[
F_1 = F_2 = \frac{GM^2}{0.04} = 1.67 \times 10^{-9}M^2 \quad \text{so the} \quad F = (1.67 \times 10^{-9}M^2)(1 + j) N
\]

6.44 Figure 6-9 shows a uniform sphere of original total mass \( M \) in which a spherical hole of diameter \( R \) has been formed. Show by a superposition argument that it attracts the mass \( m \) with a force

\[
F = \frac{GMM}{D^2} \left[ 1 - \frac{1}{8} \left( 1 - \frac{R}{2D} \right)^{-2} \right]
\]

Fig. 6-9

I The entire original sphere would exert a force

\[
F' = \frac{GMM}{D^2}
\]

The cut-out sphere, of radius \( R/2 \), would exert a force

\[
F'' = \frac{G(M/8)m}{(D - R/2)^2}
\]

By superposition \( F + F'' = F' \), or \( F = F' - F'' \), and this leads to the desired result.

6.45 Consider an attractive force which is central but inversely proportional to the first power of the distance \( (F \propto 1/r) \). Prove that if a particle is in a circular orbit under such a force, its speed is independent of the orbital radius, but its period is proportional to the radius.

I Denoting the proportionality constant by \( C \), we have \( F = -C/r \). Applying Newton's second law to a particle of mass \( m \) moving in a circular orbit with speed \( v \), we obtain \( mv^2/r = C/r \). Therefore \( v = \sqrt{C/m} \), independent of the radius \( r \). Then \( T = 2\pi v = 2\pi \sqrt{m/C} \), so the orbital period is proportional to \( r \).

6.46 A straight rod of length \( L \) extends from \( x = a \) to \( x = L + a \). Find the gravitational force it exerts on a point mass \( m \) at \( x = 0 \) if the mass per unit length of the rod is \( \mu = A + Bx^2 \).

I From Fig. 6-10,

\[
dF = \frac{Gm(\mu \text{ dx})}{x^2} \quad \text{from which} \quad F = Gm \int_0^a (A + Bx^2) \frac{dx}{x^2} = Gm \left[ \frac{1}{a} - \frac{1}{a + L} \right] + BL
\]
6.47  Repeat Prob. 6.46 if \( \mu = Ax + Bx^2 \).

\[
F = Gm \int_a^{x_{\ast} + L} \frac{(Ax + Bx^2) dx}{x} = Gm \left[ A \ln \left( 1 + \frac{L}{a} \right) + BL \right]
\]

6.48  If the earth–moon distance is \( 3.8 \times 10^4 \) km, compute the time (in days) it takes the moon to circle the earth. \( (M_s = 5.98 \times 10^{24} \text{ kg}) \)

Apply the formula of Prob. 6.36(a):

\[
T = \frac{2\pi (3.8 \times 10^4)^{3/2}}{\sqrt{(6.67 \times 10^{-11})(5.98 \times 10^{24})}} \cdot \frac{1 \text{ day}}{24 \times 3600 \text{ s}} = 27 \text{ days}
\]

6.49  A standard 1-kg mass is suspended from each side of a sensitive beam balance, as in Fig. 6-11. The wire supporting the right mass goes through an opening in the floor so that it is 10.00 m below the left mass.

(a) What is the fractional excess in the weight of the right mass over that of the left mass? (This is done most easily by using differentials.) (b) How many milligrams must be placed on the left mass to restore the balance?

(a) The magnitude of the weight force \( W \) on a mass \( m \) located at distance \( R \) from the center of the earth is given by \( W = \frac{GMm}{R^2} \). If two objects of equal mass are located at radial distances \( R_1 \) and \( R_2 = R_1 + dR \), the difference in weight is

\[
dW = W_2 - W_1 = \left( \frac{dW}{dR} \right)_{R_1} dR = \frac{-2GMm}{R_1^3} dR
\]

The fractional difference is

\[
\frac{dW}{W_1} = \frac{-2 dR}{R_1}
\]

With \( R_1 = 6.37 \times 10^6 \text{ m} \) and \( dR = -10 \text{ m} \), we find \( dW/W_1 = (20 \text{ m})/(6.37 \times 10^6 \text{ m}) = 3.14 \times 10^{-6} \). (b) To balance the excess weight force on the right-hand side in Fig. 6-11, we must increase the mass on the left by \( dm = (dW/W_1)m = (3.14 \times 10^{-6})(1 \text{ kg}) = 3.14 \text{ mg} \).

6.50  Describe the kinematics and dynamics of motion along an arbitrary plane curve.

In a general motion (Fig. 6-12), described in terms of a particle's distance \( s = s(t) \) traveled along a curved
path, the particle has

\[ \text{speed} = v = \frac{ds}{dt} \quad \text{tangential component of acceleration} = a_t = \frac{d^2s}{dt^2} \]

\[ \text{normal component of acceleration} = a_n = \frac{v^2}{\rho} = \rho \omega^2 \]

where \( \rho \) is the radius of curvature of the path and where \( \omega = v/\rho \) is defined as the particle's angular speed of rotation about an axis through the instantaneous center of curvature. Newton's second law gives \( F_n = ma_n \) and \( F_t = ma_t \), where the resultant force acting on the particle has a normal component \( F_n \) and a tangential component \( F_t \). Note that a positive \( F_n \) produces acceleration toward the center of curvature.

6.51c The angular acceleration of the toppling pole shown in Fig. 6.13 is given by \( \alpha = k \sin \theta \), where \( \theta \) is the angle between the axis of the pole and the vertical, and \( k \) is a constant. The pole starts from rest at \( \theta = 0 \). Find (a) the tangential and (b) the centripetal acceleration of the upper end of the pole in terms of \( k \), \( \theta \) and \( l \) (the length of the pole).

(a) \[ a_t = \frac{dv}{dt} = \frac{d}{dt} (l\omega) = l\alpha = lk \sin \theta \]

(b) From \( d\omega/dt = \alpha \),

\[ d\omega = k \sin \theta \, dt = k \sin \theta \frac{d\theta}{d\theta} \, d\theta = \frac{k}{\omega} \sin \theta \, d\theta \]

Then

\[ \int_0^\theta \omega \, d\omega = k \int_0^\theta \sin \theta \, d\theta \quad \text{or} \quad \omega^2 = 2k(1 - \cos \theta) \]

and \( a_n = l\omega^2 = 2kl(1 - \cos \theta) \).
6.52' Find (a) the velocity and (b) the acceleration in polar coordinates for an object moving in a curved path in a plane.

1. (a) Consider the motion of a particle along the curve \( \mathbf{R} = \mathbf{R}(t) \) shown in Fig. 6-14. At a point of the curve, the unit vectors \( \hat{r}, \hat{\theta} \) are given in terms of the unit vectors \( \hat{i}, \hat{j} \) by

\[
\hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j} \quad \hat{\theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}
\]

The velocity is given by

\[
v = \frac{d\mathbf{R}}{dt} = \frac{d(R\hat{R})}{dt} = \frac{dR}{dt} \hat{r} + R \frac{d\hat{r}}{dt}
\]

But

\[
\frac{d\hat{r}}{dt} = -\hat{i}(\sin \theta) \frac{d\theta}{dt} + \hat{j}(\cos \theta) \frac{d\theta}{dt} = \omega \hat{\theta}
\]

where \( \omega = \frac{d\theta}{dt} \), and so

\[
v = \dot{R} \hat{r} + R \omega \hat{\theta}
\]

It is seen that the velocity has a radial component \( \dot{R} \) and an angular component \( R \omega \).

2. (b)

\[
\mathbf{a} = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{dR}{dt} \hat{r} + R \frac{d\hat{r}}{dt} \right) = \frac{d^2R}{dt^2} \hat{r} + \frac{dR}{dt} \frac{d\hat{r}}{dt} + 2 \frac{dR}{dt} \frac{d\hat{r}}{dt} + R \frac{d^2\hat{r}}{dt^2}
\]

Substituting \( \frac{d\hat{r}}{dt} = \omega \hat{\theta} \) and

\[
\frac{d^2\hat{r}}{dt^2} = \frac{d(\omega \hat{\theta})}{dt} = \omega \hat{\theta} + \omega \frac{d\hat{\theta}}{dt} = \omega \hat{\theta} + \omega \left[ -\hat{i}(\cos \theta) \omega - \hat{j}(\sin \theta) \omega \right] = \omega \hat{\theta} - \omega^2 \hat{\theta}
\]

we obtain

\[
\mathbf{a} = (\ddot{R} - R \omega^2) \hat{r} + (K \alpha + 2K \omega) \hat{\theta}
\]

Fig. 6-14

6.53' A bead slides on a long bar with constant speed \( v_0 \) relative to the bar. From Fig. 6-15, \( v = \dot{r} \), where \( r \) is the distance of the bead from the axle through the end of the bar. At the same time, the bar rotates about the axle with constant angular speed \( \omega_0 \). Find (a) the velocity, (b) the acceleration, and (c) the path of the bead.

1. Use the results of Prob. 6.52 (\( R = \alpha = 0 \)).

(a)

\[
v = v_0 \hat{a} + r \omega_0 \hat{\theta}
\]

(b)

\[
a = -r \omega_0^2 \hat{a} + 2v_0 \omega_0 \hat{\theta}
\]

Fig. 6-15
(c) Integrating $\dot{r} = v_0$, $\dot{\theta} = \omega_0$, from time 0 to $t$, we get

$$r = r_0 + v_0 t \quad \theta = \theta_0 + \omega_0 t,$$

where $r_0$, $\theta_0$ are the initial values. Elimination of $t$ gives the equation of the path:

$$r - r_0 = \frac{v_0}{\omega_0} (\theta - \theta_0)$$

which is a spiral.

A Coast Guard cutter in a fog at sea is notified by radio that an illegal trawler is at a particular position $P$, 12.5 km due west of the cutter. The trawler also hears the message and heads off immediately at 12.5 km/h. The captain of the cutter anticipates this speed but does not know the direction the trawler takes. He waits for 1 h and then begins to spiral around $P$ at 48.5 km/h, with a component of velocity directed away from $P$ equal to 12.5 km/h. What is the maximum time that it takes after the message is received to catch the trawler?

From Fig. 6.16, $v_x = \frac{dr}{dt} = 12.5$ km/h

$$v_y = r \frac{d\theta}{dt} = \sqrt{(48.5)^2 - (12.5)^2} = 46.85 \text{ km/h}$$

From the first equation, $r = 12.5t$, where we have used the initial condition: $r = 12.5$ km at $t = 1$ h. Substituting for $r$ in the second equation and integrating,

$$12.5 \int_0^\theta d\theta = 46.85 \int_0^t \frac{dt}{t} \quad \theta = 3.75 \ln t$$

The spiral path of the cutter must cross the radial path of the trawler at some moment, $t = \tau$, during the first revolution. At that moment, both ships will be at the same distance from $P$, so the cutter will have indeed caught the trawler. Since $\theta \approx 2\pi$ for $t = \tau$, $3.75 \ln \tau \leq 2\pi$, or $\tau \leq e^{2\pi/3.75} = 5.34$ h.

A wet open umbrella is held upright as shown in Fig. 6.17(a) and is twirled about the handle at a uniform rate of 21 rev in 44 s. If the rim of the umbrella is a circle 1 m in diameter, and the height of the rim above the floor is 1.5 m, find where the drops of water spun off the rim hit the floor.

The angular speed of the umbrella is

$$\omega = \frac{21 \times 2\pi \text{ rad}}{44 \text{ s}} = 3 \text{ rad/s}$$

Then the tangential speed of the water drops on leaving the rim of the umbrella is $v_0 = r\omega = (0.5)(3) = 1.5 \text{ m/s}$.

To calculate the time for a drop to reach the floor use $h = \frac{1}{2}gt^2$:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(1.5)}{9.8}} = 0.553 \text{ s}$$
The horizontal range of the drop is then

\[ r = v_0 t = (1.5)(0.55) = 0.83 \text{ m}; \]

and the locus of the drops is a circle of radius

\[ R = \sqrt{(0.5)^2 + (0.83)^2} = 0.97 \text{ m}. \]

In Fig. 6.18, as the block descends, the rigid rotor winds up on its rope and, thus, ascends. Find the relations between the linear and angular accelerations and between the linear and angular speeds.

If \( \theta \) is the angle through which the rotor has turned from its initial position,

\[ y_1 = y_{10} - r\theta \quad y_2 = y_{20} + R\theta - r\theta \]

since the length of rope that is wound on the smaller cylinder is \( r\theta \) and the length of rope that is unwound from the larger cylinder is \( R\theta \). Take first and second time-derivatives of \( y_1 \) and \( y_2 \):

\[ v_1 = \dot{y}_1 = -r\theta = -r\omega \quad v_2 = \dot{y}_2 = (R - r)\theta = (R - r)\omega \]

\[ a_1 = \ddot{y}_1 = -r\phi = -r\alpha \quad a_2 = \ddot{y}_2 = (R - r)\phi = (R - r)\alpha \]

Figure 6-19 shows a ray of light that passes from air into water. The ray is bent upon passing into the water, according to Snell's law \((\sin \theta = n \sin \psi)\). The angle \( \theta \) increases at a constant rate of 10 rad/s, and \( n = 1.3 \). Find the angular speed \( \omega \) and the angular acceleration \( \alpha \) of the refracted ray for \( \theta = 30^\circ \).
MOTION IN A PLANE II  

Take first and second time-derivatives of \( \sin \theta = n \sin \psi \) to get \( \omega = \dot{\psi} \) and \( \alpha = \ddot{\psi} \); recall that \( \ddot{\theta} = 0 \).

\[
\dot{\theta} \cos \theta = n \dot{\psi} \cos \psi \quad \text{or} \quad \dot{\psi} = \frac{\dot{\theta} \cos \theta}{n \cos \psi} = \frac{\dot{\theta} \cos \theta}{\sqrt{n^2 - \sin^2 \theta}}
\]

\[-\dot{\theta}^2 \sin \theta = n \dot{\psi} \cos \psi - n \ddot{\psi} \sin \psi \quad \text{or} \quad \ddot{\psi} = \frac{n \dot{\psi}^2 \sin \psi - \dot{\theta}^2 \sin \theta}{n \cos \psi} = \frac{(\dot{\psi}^2 - \dot{\theta}^2) \sin \theta}{\sqrt{n^2 - \sin^2 \theta}}
\]

Substituting the data,

\[
\dot{\psi} = \frac{10(\sqrt{3}/2)}{\sqrt{(1.3)^2 - (1/2)^2}} = 7.22 \text{ rad/s} \quad \ddot{\psi} = \frac{[(7.22)^2 - (10)^2](1/2)}{\sqrt{(1.3)^2 - (1/2)^2}} = -20.0 \text{ rad/s}^2
\]

Fig. 6-19

Fig. 6-20

6.58° A rod leans against a stationary cylindrical body as shown in Fig. 6-20, and its right end slides to the right on the floor with a constant speed \( v \). Find (a) the angular speed \( \omega \) and (b) the angular acceleration \( \alpha \), in terms of \( v \), \( x \), and \( R \).

(a) From the geometry, \( x = R \sin \theta \). Also, \( \omega = -\dot{\theta} \). Therefore,

\[
v = \dot{x} = \frac{d}{dt} \left( \frac{R}{\sin \theta} \right) = -\frac{R \dot{\theta} \cos \theta}{\sin^2 \theta} = \frac{\omega R \sin \theta}{\sin^2 \theta} \quad \omega = v \frac{\sin^2 \theta}{R \cos \theta} = \frac{Rv}{\sqrt{x^2 - R^2}}
\]

(b)

\[
\alpha = \ddot{\omega} = \frac{d}{dt} \left( \frac{Rv}{\sqrt{x^2 - R^2}} \right) = \frac{-Rv^2 (2x^2 - R^2)}{x^2 (x^2 - R^2)^{3/2}}
\]

6.59° A particle of mass \( m \) moves without friction along a semicubical parabolic curve, \( y^3 = ax^4 \), with constant speed \( v \). Find the reaction force of the curve on the particle.

The local radius of curvature of the curve is

\[
\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (9/4ax^3)]^{3/2}}{(3/4)a^{1/2}x^{-1/2}}
\]

In this motion of the particle, the curve exerts a normal or centripetal force, causing the particle momentarily to move in an arc of a circle of radius \( \rho \) (see Fig. 6-12). Thus,

\[
F = \frac{mv^2}{\rho} = \frac{3}{4} a^{1/2} \rho^{-1/2} \left( 1 + \frac{9}{4} ax \right)^{-3/2} mv^2
\]

6.60° A particle whose mass is 2 kg moves with a speed of 44 m/s on a curved path. The resultant force acting on the particle at a particular point of the curve is 30 N at 60° to the tangent to the curve, as shown in Fig. 6-21. At that point, find (a) the radius of curvature of the curve and (b) the tangential acceleration of the particle.

(a)

\[
\rho = \frac{mv^2}{F_x} = \frac{2(44)^2}{30 \sin 60°} = 149 \text{ m}
\]
6.61* A bug is crawling with constant speed \( v \) along the spoke of a bicycle wheel, of radius \( a \), while the bicycle moves down the road with constant speed \( V \). Find the accelerations of the bug, as observed by a man standing beside the road, along the perpendicular to the spoke of the wheel.

Choose a coordinate system that travels with the center of the wheel; accelerations in this coordinate system are the same as in the ground system, because the two systems have a constant relative velocity. Applying the results of Prob. 6.52 with \( \omega = V/a \), \( \vec{R} = \vec{v} \), we find

\[
a_r = -\frac{\vec{R} \cdot \vec{V}^2}{\vec{a}^2} = 2v \frac{V}{a}
\]

\[
a_o = 2v \frac{V}{a}
\]