Sturm-Liouville Theory
(boundary value)

A second order Sturm–Liouville problem is a homogeneous boundary value problem of the form

\[
[P(x) y']' + Q(x) y + \lambda w(x) y = 0 \\
\alpha_1 y(a) + \beta_1 y'(a) = 0 \\
\alpha_2 y(b) + \beta_2 y'(b) = 0
\]

where \( P, P', Q, w \) are continuous and real on \([a, b]\), and \( P \) and \( w \) are positive.

Theorem: For \( y_1 \) and \( y_2 \) two linearly independent solutions of the homogeneous differential equation, nontrivial solutions of the homogeneous boundary value problem exist iff

\[
\det \begin{vmatrix}
\alpha_1 y_1(a) + \beta_1 y_1'(a) & \alpha_1 y_2(a) + \beta_1 y_2'(a) \\
\alpha_2 y_1(b) + \beta_2 y_1'(b) & \alpha_2 y_2(b) + \beta_2 y_2'(b)
\end{vmatrix} = 0
\]

Definition: Values of \( \lambda \) for which nontrivial solutions exist are called eigenvalues. The corresponding solutions are called eigenfunctions.
Theorem: The eigenvalues of a homogeneous Sturm-Liouville problem are real and non-negative and can be arranged in a strictly increasing infinite sequence

\[ 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots \]

and \( \lambda_n \to \infty \) as \( n \to \infty \).

Theorem: For each eigenvalue, there exists exactly one linearly independent eigenfunction, \( y_n \). These eigenfunctions for differing eigenvalues are orthogonal with respect to the inner product:

\[
(y_n, y_m)_w = \int_a^b y_n(x) y_m(x) w(x) \, dx = N_n \delta_{n,m}
\]

Theorem: The eigenfunctions \( y_n \) span the vector space of piecewise smooth functions satisfying the boundary conditions of the Sturm-Liouville problem. (Convergence in the mean, not pointwise.)

\[
f(x) = \sum_{n=1}^{\infty} c_n y_n(x)
\]

where the \( c_n \)'s are given by:

\[
c_n = \frac{1}{N_n} (y_n, f)_w = \frac{1}{N_n} \int_a^b y_n^*(x) f(x) w(x) \, dx
\]
A second order periodic Sturm–Liouville problem is a homogeneous problem of the form

\[ [P(x) y']' + Q(x) y + \lambda w(x) y = 0 \]

where \( P, P', Q, w \) are continuous and real on \([a, b] \), and \( P \) and \( w \) are positive, and

\[ \left[ P(x) \left( f^* (x) g'(x) - f^*(x) g(x) \right) \right] \bigg|_{a}^{b} = 0 \]

for \( f(x) \) and \( g(x) \) and two vectors in the vector space.

Definition: Values of \( \lambda \) for which nontrivial solutions of the periodic Sturm-Liouville problem exist are called eigenvalues. The corresponding solutions are called eigenfunctions.

Theorem: The eigenvalues of a periodic Sturm-Liouville problem are real.

Theorem: For each eigenvalue, there exist linearly independent eigenfunctions, \( y_n \). These eigenfunctions for differing eigenvalues are orthogonal with respect to the
inner product:

$$(y_n, y_m)_w = \int_a^b y_n(x) y_m(x) w(x) \, dx = N_n \delta_{n,m}$$

Eigenfunctions with the same eigenvalue can be orthogonalized using Gram-Schmidt orthogonalization.

Theorem: The eigenfunctions $y_n$ span the vector space of piecewise smooth functions satisfying the boundary conditions of the Sturm-Liouville problem. (Convergence in the mean, not pointwise.)

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

where the $c_n$’s are given by:

$$c_n = \frac{1}{N_n} (y_n, f)_w = \frac{1}{N_n} \int_a^b y_n^*(x) f(x) w(x) \, dx$$