Legendre Functions and Legendre Polynomials

Legendre functions are solutions of the Legendre ODE:

\[
\left(1 - z^2\right)y'' - 2zy' + \ell(\ell + 1)y = 0 \quad \text{where} \quad \ell \text{ is a real number.}
\]

**Series solutions:**

\[
y_1(z) = 1 - \frac{\ell(\ell + 1)}{2!}z^2 + \frac{\ell(\ell + 1)(\ell - 2)(\ell + 3)}{4!}z^4 + \ldots
\]

\[
y_2(z) = z - \frac{(\ell - 1)(\ell + 2)}{3!} + \frac{(\ell - 1)(\ell + 2)(\ell - 3)(\ell + 4)}{5!}z^4 + \ldots
\]

For integer \( \ell = 0, 1, 2, 3, \ldots \), one of series \( y_1(z) \) or \( y_2(z) \) terminates to yield a polynomial. The other solution is an infinite series \( Q_\ell(z) \), convergent for \( |z| < 1 \), known as a Legendre function of the 2\(^{nd}\) kind.

**Legendre polynomials:**

For \( \ell \) even, the series for \( y_1 \) yields polynomials \( P_0(z), P_2(z), P_4(z), \ldots \)
For \( \ell \) odd, the series for \( y_2 \) yields polynomials \( P_1(z), P_3(z), P_5(z), \ldots \)

The first six Legendre polynomials, normalized so that \( P_\ell(1) = 1 \), are:

\[
P_0(z) = 1 \quad P_1(z) = z
\]

\[
P_2(z) = \frac{1}{2}(3z^2 - 1) \quad P_3(z) = \frac{1}{2}(5z^3 - 3z)
\]

\[
P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3) \quad P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z)
\]

**Recurrence relations:**

\[
P_\ell = P_{\ell+1}' - 2zP'_\ell + P'_{\ell-1} \quad (\ell + 1)P_\ell = P_{\ell+1}' - zP'_\ell
\]

\[
(\ell + 1)P_{\ell+1}' - (2\ell + 1)zP_\ell + \ell P_{\ell-1} = 0
\]

**Generating function:**

\[
G(z, h) = \left(1 - 2zh + h^2\right)^{1/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n}{dh^n} \right)_{h=0} h^n = \sum_{n=0}^{\infty} P_n(z)h^n
\]
Orthogonality and Normalization:

$$\int_{-1}^{1} P_\ell(z) P_k(z) dz = \begin{cases} 0 & \text{if } \ell \neq k \\ \frac{2}{2\ell + 1} & \text{if } \ell = k \end{cases}$$

Legendre functions of the 2<sup>nd</sup> kind:

- \( \ell \) even: \( Q_\ell(z) = \frac{(-1)^{\ell/2}}{\ell!} 2^\ell \left\{ (\ell/2)! \right\}^2 \cdot y_2(z) \) \quad \text{with} \quad Q_0 = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right) 
- \( \ell \) odd: \( Q_\ell(z) = \frac{(-1)^{\ell+1/2}}{\ell!} 2^\ell \left\{ (\ell-1)/2 \right\}^2 \cdot y_1(z) \) \quad \text{with} \quad Q_1 = \frac{1}{2} z \ln \left( \frac{1+z}{1-z} \right) 

Recurrence relations for \( Q_\ell(z) \) are the same as for \( P_\ell(z) \).
**Bessel Functions**

Bessel functions are solutions of Bessel's ODE:

\[ z^2 y'' + zy' + \left(z^2 - \nu^2\right)y = 0 \quad \text{where } \nu \geq 0. \]

**Solutions for non-integer \( \nu \):**

Linearly independent solutions are \( J_\nu(z) \) and \( J_{-\nu}(z) \) where

\[
J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{\nu + 2n}
\]

Eq. (1).

The functions \( J_\nu(z) \) and \( J_{-\nu}(z) \) are known as **Bessel functions of the 1\text{st} kind**.

\[ \Gamma(n) = \int_0^\infty x^{n-1}e^{-x} \, dx \]

is a **Gamma function**. Note that \( \Gamma(1/2) = \sqrt{\pi} \) and \( \Gamma(n+1) = n\Gamma(n) \).

\[ J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{and} \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z \]

**Solutions for integer \( \nu \):**

The first solution for \( \nu = 0 \) is \( J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2^{2n} n! \Gamma(1+n)} \).

For \( \nu = 1, 2, 3, 4, \ldots \) the first solution is \( J_\nu(z) \) given by Eq. (1) above.

The second solution is

\[
Y_\nu(z) = \lim_{\mu \to \nu} \left[ \frac{J_\mu(z) \cos \nu \pi - J_\nu(z)}{\sin \mu \pi} \right]
\]

The functions \( Y_\nu(z) \) are known as **Bessel functions of the 2\text{nd} kind**.

**Recurrence relations:**

\[
\frac{d}{dz} \left[ z^\nu J_\nu(z) \right] = z^\nu J_{\nu-1}(z) \quad \quad \quad \quad \frac{d}{dz} \left[ z^{-\nu} J_\nu(z) \right] = -z^{-\nu} J_{\nu+1}(z)
\]

\[
J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)
\]
Generating function (Bessel functions of the 1st kind, integer $\nu$):

$$G(z,h) = \exp\left[\frac{Z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)h^n$$

Orthogonality and Normalization (Bessel functions of the 1st kind):

In an interval $[a, b]$ such that $J_{\nu}(\lambda a) = J_{\nu}(\mu a) = J_{\nu}(\lambda b) = J_{\nu}(\mu b) = 0$

OR $J'_{\nu}(\lambda a) = J'_{\nu}(\mu a) = J'_{\nu}(\lambda b) = J'_{\nu}(\mu b) = 0$

$$\int_{a}^{b} z J_{\nu}(\lambda z) J_{\nu}(\mu z) dz = 0 \quad \text{if } \lambda \neq \mu$$

$$= \frac{1}{2}\left[\left(z^2 - \frac{\nu^2}{\lambda^2}\right) J_{\nu}^2(\lambda z) + z^2 J'_{\nu}(\lambda z)^2\right]_{a}^{b} \quad \text{if } \lambda = \mu$$