The Algebra of Complex Numbers

Complex numbers are algebraic expressions containing the factor $i \equiv \sqrt{-1}$. A complex number z consists of a **"real" part**, Re $z \equiv x$, and an **"imaginary" part**, Im $z \equiv y$, that is,

$$z = \operatorname{Re} z + i \operatorname{Im} z = x + iy .$$

If Im z = 0, then z = x is a "real number". If Re z = 0, then z = iy is said to be "purely imaginary." The real and imaginary parts, x and y, are themselves real numbers.

Addition and subtraction: $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

Multiplication: $z_1 \times z_2 = (x_1 + iy_1) \times (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Remember! $i^2 = -1$.

Division: $\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

Modulus and Argument:

The **modulus** of a complex number is defined by $|z| = \sqrt{x^2 + y^2}$. (Sometimes called the "absolute value.")

The **argument** is $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$.

Complex Conjugate:

The **complex conjugate** is defined by $z^* = x - iy$. The following rules apply:

$$zz^* = |z|^2$$

$$(z_1 + z_2)^* = z_1^* + z_2^*$$

$$(z_1 z_2)^* = z_1^* z_2^*$$

$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$$

Some useful relations involving complex conjugates:

$$\left(z^*\right)^* = z$$
$$z + z^* = 2\,\mathrm{Re}\,z = 2x$$
$$z - z^* = 2i\,\mathrm{Im}\,z = 2iy$$
$$\frac{z}{z^*} = \left(\frac{x^2 - y^2}{x^2 + y^2}\right) + i\left(\frac{2xy}{x^2 + y^2}\right)$$

Polar representation of complex numbers:

$z = re^{i\theta}$ with r and θ real. $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$.

Euler relations: $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$.

$e^{2n\pi i} = 1$ $e^{i\pi/2} = i$ $e^{i\pi} = -1$ $e^{i3\pi/2} = -i$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$ $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Argand Diagram:

A complex number $z = x + iy$ is represented as a point in the plane Im z – Re z or, equivalently, as a vector with components x and y. The length of the vector is the modulus $r = |z|$, and the direction is determined by the angle $\theta = \arg z$.