PH 422: Day 4

1 The Divergence Theorem

The total flux of the electric field out through a small rectangular box is

$$\text{flux} = \sum_{\text{box}} \vec{E} \cdot \vec{A} = \nabla \cdot \vec{E} \ dV$$

But any closed box can be filled with such boxes. Furthermore, the flux out of such a box is just the sum of the flux out of each of the smaller boxes, since the net flux through any common side will be zero (because adjacent boxes have opposite notions of “out of”). Thus, the total flux out of any closed box is given by

$$\int_{\text{box}} \vec{E} \cdot \vec{A} = \int_{\text{inside}} \nabla \cdot \vec{E} \ dV$$

This is the Divergence Theorem.

2 Differential form of Gauss’ Law

Recall that Gauss’ Law says that

$$\int_{\text{box}} \vec{E} \cdot \vec{A} = \frac{1}{\varepsilon_0} Q_{\text{inside}}$$

But the enclosed charge is just

$$Q_{\text{inside}} = \int_{\text{box}} \rho \ dV$$

so we have

$$\int_{\text{box}} \vec{E} \cdot \vec{A} = \frac{1}{\varepsilon_0} \int_{\text{box}} \rho \ dV$$

Putting this all together, the Divergence Theorem tells us that

$$\int_{\text{inside}} \nabla \cdot \vec{E} \ dV = \frac{1}{\varepsilon_0} \int_{\text{box}} \rho \ dV$$
for any closed box. This means that the integrands themselves must be equal, that is,

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

This is the differential form of Gauss’ Law, and is one of Maxwell’s Equations. It states that the divergence of the electric field at any point is just a measure of the charge density there.

3 The Divergence in Curvilinear Coordinates

The first expression above relating flux to divergence is really just a definition of the divergence of the electric field; from this point of view, the Divergence Theorem is a tautology. Our computations yesterday used this geometric definition to derive an expression for \( \nabla \cdot \vec{E} \) in rectangular coordinates, namely

\[ \nabla \cdot \vec{E} = \frac{\text{flux}}{\text{unit volume}} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]

Similar computations to those done yesterday in rectangular coordinates can be done using boxes adapted to other coordinate systems. Not surprisingly, this introduces some additional factors of \( r \) (and \( \sin \theta \)). For instance, consider a radial vector field of the form

\[ \vec{E} = E_r \hat{r} \]

where \( \hat{r} \) is the unit vector in the radial direction. The electric field of a point charge would have this form, as do the spherical examples some groups considered in a previous activity. What is the flux of \( \vec{E} \) through a small box around an arbitrary point \( P \), whose sides are surfaces with one of the spherical coordinates held constant? Only the two sides which are parts of spheres contribute, and each such contribution takes the form

\[ \vec{E} \cdot d\vec{A} = E_r r^2 \sin \theta d\theta d\phi \]

An argument similar to the one used in rectangular coordinates leads to

\[ \vec{E} \cdot d\vec{A} = \frac{\partial}{\partial r} \left( r^2 E_r \right) \sin \theta dr d\theta d\phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 E_r \right) dV \]
where it is important to note that the factor of \( r^2 \) must also be differentiated. It now finally follows that

\[
\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 E_r \right)
\]

This is just part of the full expression for the divergence in spherical coordinates, since we started with the very special vector field \( \vec{E} \). These formulas are so important that they appear on the inside front cover of Griffiths.

### 4 The Divergence of a Coulomb Field

The electric field of a point charge at the origin is given by

\[
\vec{E} = \frac{1}{4\pi \epsilon_0} \frac{q \hat{r}}{r^2}
\]

We can take the divergence of this field using the expression derived above for the divergence of a radial vector field, which yields

\[
\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 E_r \right) = \frac{1}{4\pi \epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{q r}{r^2} \right) = 0
\]

On the other hand, the flux of this electric field through a sphere centered at the origin is

\[
\int_{\text{sphere}} \vec{E} \cdot d\vec{A} = \int_{\text{sphere}} \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} d\vec{A} = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \left( 4\pi r^2 \right) = \frac{q}{\epsilon_0}
\]

in agreement with Gauss' Law. The Divergence Theorem then tells us that

\[
\int_{\text{inside}} \nabla \cdot \vec{E} \, dV = \int_{\text{sphere}} \vec{E} \cdot d\vec{A} \neq 0
\]

even though \( \nabla \cdot \vec{E} = 0 \). What’s going on?

A bit of thought yields a clue: \( \vec{E} \) isn’t defined at \( r = 0 \); neither is its divergence. So we have a function which vanishes almost everywhere, whose integral isn’t zero. This should remind you of the Dirac delta function. However, we’re in 3 dimensions here, so that the correct conclusion is

\[
\nabla \cdot \frac{1}{4\pi \epsilon_0} \frac{q \hat{r}}{r^2} = \frac{q}{\epsilon_0} \delta^3(\vec{r}) = \frac{q}{\epsilon_0} \delta(x) \delta(y) \delta(z)
\]

or equivalently

\[
\rho = q \delta^3(\vec{r})
\]

which should not be surprising.