PH 422: Day 3

1 Electric Field Lines

You probably already know the following facts about electric field lines:

- The density of field lines is proportional to the strength of the electric field there;
- Field lines only start at positive charges and end at negative charges;
- Field lines never cross.

These rules can all be explained using Gauss’ Law, since the flux of the electric field can be interpreted as counting the number of field lines which cross the surface.

2 Divergence

Consider a small closed box, with sides parallel to the coordinate planes. What is the flux of $\vec{E}$ out of the box?

Consider first the vertical contribution, namely the flux up through the top plus the flux down through the bottom. These two sides each have area element $dA = dx\,dy$, but the outward normal vectors point in opposite directions, so we get

$$\sum_{\text{top+bottom}} \vec{E} \cdot d\vec{A} = \vec{E}(z + dz) \cdot \hat{k} \, dx \, dy - \vec{E}(z) \cdot \hat{k} \, dx \, dy$$

$$= \left( E_z(z + dz) - E_z(z) \right) \, dx \, dy$$

$$= \frac{E_z(z + dz) - E_z(z)}{dz} \, dx \, dy \, dz$$

$$= \frac{\partial E_z}{\partial z} \, dx \, dy \, dz$$

Repeating this argument using the remaining pairs of faces, it follows that the total flux out of the box is

$$\text{total flux} = \sum_{\text{box}} \vec{E} \cdot d\vec{A} = \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \, d\tau$$
Since this is proportional to the volume of the box, it approaches zero as the box shrinks down to a point. The interesting quantity is therefore the ratio of the flux to volume. This is the divergence.

At any point \( P \), we therefore define the divergence of a vector field \( \vec{E} \), written \( \nabla \cdot \vec{E} \), to be the flux of \( \vec{E} \) per unit volume leaving a small box around \( P \). In other words, the divergence is the limit as the box collapses around \( P \) of the ratio of the flux of the electric field out of the box to the volume of the box. Thus, the divergence of \( \vec{E} \) at \( P \) is the flux per unit volume through a small box around \( P \), which is given in rectangular coordinates by

\[
\nabla \cdot \vec{E} = \frac{\text{flux}}{\text{unit volume}} = \frac{\partial E_x}{\partial x} \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}
\]

You may have seen this formula before, but remember that it is merely the rectangular coordinate expression for the divergence of \( \vec{E} \); the divergence is defined as flux per unit volume. Similar computations can be used to determine expressions for the divergence in other coordinate systems.