1 The Electric field of a uniformly charged plane

Recall that the electric field of a uniform disk is given along the axis by

$$\vec{E}(z) = \frac{2\pi \sigma \hat{z}}{4\pi \epsilon_0} \left( \frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{z^2 + R^2}} \right)$$

where of course $\frac{z}{\sqrt{z^2}} = \pm 1$ depending on the sign of $z$. In the limit as $R \to \infty$, one gets the electric field of a uniformly charged plane, which is just

$$\vec{E}(z) = \text{sgn}(z) \frac{\sigma \hat{z}}{2\epsilon_0}$$

which is valid everywhere, as any point can be thought of as being on the axis. But the calculation leading to the first expression above was somewhat involved. Is there a better way?

An infinite plane is highly symmetric. Not only is every point like every other, at any point all directions in the plane are equivalent. This rotational symmetry means that the electric field must be orthogonal to the plane — otherwise, you could face a different direction, repeat the computation, and get a different answer. (There is a subtlety here, in that the electric field must in fact point in opposite directions on opposite sides of the plane.) A similar argument using the translational symmetry shows that the electric field can only depend on the distance from the plane. The electric field must therefore be of the form

$$\vec{E} = E(z) \hat{z}$$

assuming that the $z$-direction is orthogonal to the plane.

Now recall Gauss' Law, which relates the flux of the electric field through any closed surface to the charge enclosed by the surface, that is,

$$\int \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Choose a closed surface which uses the symmetry. A rectangular box is one possibility, two of whose faces are parallel to the plane, and equidistant from it. The flux through the sides of this box is zero, since the normal vector to these sides is parallel to the plane, but $\vec{E}$ is perpendicular to the plane, so
that $\mathbf{E} \cdot d\mathbf{A} = 0$. What about the top? The electric field is perpendicular to the top, but constant in magnitude. Thus,

$$\int_{\text{top}} \mathbf{E} \cdot d\mathbf{A} = E(z)A$$

where $z$ is the distance from the plane to the top of the box (where $z > 0$). By symmetry, the flux through the bottom of the box is the same as that though the top. (Point out that this implies that $E(-z) = -E(z)$.) Finally, the charge enclosed by the box is just the charge density $\sigma$ times the area of the part of the plane inside the box, which is again $A$. Inserting all of this into Gauss’ Law, we obtain

$$2E(z)A = \frac{\sigma A}{\epsilon_0}$$

so that

$$E(z) = \frac{\sigma}{2\epsilon_0}$$

which turns out to be independent of $z$ (for $z > 0$).