PH 422: Day 13

1 Electrostatic Energy

How much work is done in assembling a collection of point charges? Work is force times distance, which in this context takes the form

\[ W = \int \vec{F} \cdot d\vec{r} = -q \int \vec{E} \cdot d\vec{r} = q \Delta V \]

Moving the first charge requires no work — since there is no electric field. The second charge needs to be moved in the (Coulomb) field of the first, the third in the field of the first two, and so on. Continuing in this manner, we see that the work done in assembling the charges is

\[ W = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{8\pi\epsilon_0} \sum_{i\neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} \]

The advantage of the second expression, in which each term is double-counted, is that it can be rewritten in the form

\[ W = \frac{1}{2} \sum q_i V(\vec{r}_i) \]

where \( V(\vec{r}_i) \) is the potential at the location \( \vec{r}_i \) of the \( i \)th charge due to all the other charges. This expression in turn generalizes naturally to a continuous charge distribution (but see the discussion in Griffiths about some subtleties in this limit), for which it becomes

\[ W = \frac{1}{2} \int \rho V d\tau \]

Now use Gauss’ Law, to replace \( \rho \) by \( (\epsilon_0 \times \text{times}) \) the divergence of the electric field. We have

\[ W = \frac{\epsilon_0}{2} \int (\nablaslash \cdot \vec{E}) V d\tau \]

\[ = \frac{\epsilon_0}{2} \left( -\int \vec{E} \cdot \nablaslash V d\tau + \int V \vec{E} \cdot d\vec{A} \right) \]

\[ = \frac{\epsilon_0}{2} \left( \int |\vec{E}|^2 d\tau + \int V \vec{E} \cdot d\vec{A} \right) \]

where we have integrated by parts, as justified below.
2 Integration by parts

The product rule for the divergence is

$$\nabla \cdot (f \vec{G}) = (\nabla f) \cdot \vec{G} + f (\nabla \cdot \vec{G})$$

Integrating both sides yields

$$\int \nabla \cdot (f \vec{G}) \, d\tau = \int (\nabla f) \cdot \vec{G} \, d\tau + \int f (\nabla \cdot \vec{G}) \, d\tau$$

Now use the Divergence Theorem to rewrite the first term, leading to

$$\int (f \vec{G}) \cdot d\vec{A} = \int (\nabla f) \cdot \vec{G} \, d\tau + \int f (\nabla \cdot \vec{G}) \, d\tau$$

which can be rearranged to

$$\int f (\nabla \cdot \vec{G}) \, d\tau = \int (f \vec{G}) \cdot d\vec{A} - \int (\nabla f) \cdot \vec{G} \, d\tau$$

which is the desired integration by parts.