PH 422: Day 1

1 Flux

Recall that

\[ V = -\int \vec{E} \cdot d\vec{r} \]

This is the integral of \( \vec{E} \) along a curve. What sort of integral can you take of \( \vec{E} \) over a surface? Along a curve, there is a natural direction, namely that tangent to the curve. The line integral above adds up the tangential component of \( \vec{E} \) along the curve. The only natural direction associated with a surface, on the other hand, is the direction perpendicular to it. The natural integral to compute over a surface adds up the normal component of \( \vec{E} \). This is called the \textit{flux} of \( \vec{E} \) through the given surface (in the given direction):

\[ \text{flux} = \int \vec{E} \cdot \hat{n} \, dA = \int \vec{E} \cdot d\vec{A} \]

Be warned that some authors use two integral signs rather than one, and other letters are often used in place of \( A \).

2 Gauss’ Law

Recall Gauss’ Law, which says that

\[ \int_{\text{box}} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} Q_{\text{inside}} \]

Consider the example of a point charge inside a sphere, where

\[ \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q\hat{r}}{r^2} \]

and

\[ d\vec{A} = r^2 \sin \theta \, d\theta \, d\phi \, \hat{r} \]
Thus, the flux is given by

$$\int_{\text{sphere}} \vec{E} \cdot d\vec{A} = \int_{\text{sphere}} \frac{1}{4\pi\epsilon_0} \frac{q\hat{r}}{r^2} \cdot \hat{r} dA$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \times (\text{surface area})$$

$$= \frac{q}{\epsilon_0}$$

as claimed. Note that there is no dependence on $r$.

3 Example: Flux through a cube

Gauss’ Law does not depend on the shape of the surface being used. So let’s replace the sphere in the previous example with a cube. Suppose the charge is at the origin, and the length of each side of the cube is 2. What is the flux through one face? We have

$$\text{flux} = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{4\pi\epsilon_0} \frac{q\hat{r}}{r^2} \cdot \hat{k} \, dx \, dy = \frac{q}{4\pi\epsilon_0} \int_{-1}^{1} \int_{-1}^{1} \frac{dx \, dy}{\sqrt{x^2 + y^2 + 1}}$$

This integral is doable with the help of integral tables or with Maple. The worksheet flux.mws can be used to compute the answer. Does it agree with the previous computation for the sphere?

Now suppose the charge is not at the origin. Maple can still do the integrals numerically; try some examples.

Finally, suppose the charge is on a face, or an edge. What answer do you expect? Check and see.