The Dirac Delta Function

The delta function is not a function. It is a mathematical entity called a distribution which is well defined only when it appears under an integral sign. However, you can think of it as a generalization of a function with the following defining properties:

\[
\delta(x) = \begin{cases} 
0, & \text{if } x \neq 0 \\
\infty, & \text{if } x = 0 
\end{cases}
\]

\[
\int_{b}^{c} \delta(x) \, dx = 1 \quad \text{for } b < 0 < c
\]

Another function which we will use in this review is the theta, Heaviside, or step function. It is a true function, but is discontinuous. It is defined by:

\[
\Theta(x) = \begin{cases} 
1, & x > 0 \\
0, & x < 0 
\end{cases}
\]

The value of \(\Theta(0)\) is a matter of convention. Usually you will not need to choose it. If you do, choosing the average value \(\Theta(0) = 1/2\) is safest.

The three dimensional delta function \(\delta^3(\vec{r})\) is just the product of three one dimensional delta functions:

\[
\delta^3(\vec{r}) = \delta(x) \delta(y) \delta(z)
\]

where \(\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}\) is the position vector. It satisfies:

\[
\int_{all \ space} \delta^3(\vec{r}) \, d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) \delta(z) \, dx \, dy \, dz = 1
\]
There are many properties of the delta function which follow from the properties above. Some of these are:

1. \( \delta(x) = \delta(-x) \)
2. \( \delta'(x) = -\delta'(-x) \) where \( \delta \) is derivative w.r.t. \( x \)
3. \( \Theta'(x) = \delta(x) \)
4. \( \int_b^c f(x) \delta(x - a) \, dx = f(a) \)
5. \( \int_b^c f(x) \delta'(x - a) \, dx = -f'(a) \)
6. \( \delta(ax) = \frac{1}{|a|} \delta(x) \) for constant \( a \)
7. \( \delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i) \) where \( g(x_i) = 0 \) and \( g'(x_i) \neq 0 \)
8. \( \delta(x^2 - a^2) = |2a|^{-1} \left[ \delta(x - a) + \delta(x + a) \right] \)
9. \( \delta((x - a)(x - b)) = \frac{1}{|a - b|} \left[ \delta(x - a) + \delta(x - b) \right] \)
10. \( \int \text{all space} \ f(\vec{r}) \delta^3(\vec{r} - \vec{r}_0) \, d\tau = f(\vec{r}_0) \)
Some useful representations of the delta function are:

\[\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} dt\] \hspace{1cm} (1)

\[\delta(x) = \lim_{a \to 0} \frac{1}{2a} [\Theta(x + a) - \Theta(x - a)]\] \hspace{1cm} (2)

\[\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi \epsilon}} \exp \left( -\frac{x^2}{\epsilon} \right)\] \hspace{1cm} (3)

\[\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon^2 + x^2}\] \hspace{1cm} (4)

\[\delta(x) = \lim_{N \to \infty} \frac{\sin Nx}{\pi x}\] \hspace{1cm} (5)

\[\delta(x) = \frac{1}{2} \frac{d^2}{dx^2} |x|\] \hspace{1cm} (6)

\[\delta(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t(t-x)}\] \hspace{1cm} Cauchy-Principal Value integration implied. \hspace{1cm} (7)

The closure relation is given by:

\[\delta(x - y) = \sum_{n=0}^{\infty} \phi_n(x) \phi_n(y)\] \hspace{1cm} (8)

where the \(\phi_n\) are a complete set of real orthonormal eigenfunctions for a hermitian differential operator.
The Three-Dimensional Delta Function

The three-dimensional delta function must satisfy:

\[
\int_{all \ space} \delta^3(\vec{r} - \vec{r}_0) \, d\tau = 1
\]

In rectangular coordinates:

\[
\delta^3(\vec{r} - \vec{r}_0) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0)
\]

But in curvilinear coordinates, \(d\tau = h_u h_v h_w \, du \, dv \, dw\), i.e. it has a Jacobian in it. Thus,

\[
\delta^3(\vec{r} - \vec{r}_0) \neq \delta(u - u_0) \delta(v - v_0) \delta(w - w_0)
\]

but instead:

\[
\delta^3(\vec{r}) = \frac{1}{h_u h_v h_w} \delta(u - u_0) \delta(v - v_0) \delta(w - w_0)
\]

In particular, in cylindrical coordinates:

\[
\delta^3(\vec{r}) = \frac{1}{r} \delta(r - r_0) \delta(\theta - \theta_0) \delta(z - z_0)
\]

and in spherical coordinates:

\[
\delta^3(\vec{r}) = \frac{1}{r^2 \sin(\theta)} \delta(r - r_0) \delta(\theta - \theta_0) \delta(\phi - \phi_0)
\]