PH 320: Day 7

1 Fourier series

Remarkably, any periodic function can be expanded in the form

\[ f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \left( \frac{2m\pi x}{L} \right) + \sum_{m=1}^{\infty} b_m \sin \left( \frac{2m\pi x}{L} \right) \]

How does one determine the coefficients \(a_m\) and \(b_m\)? A clue is provided by the fact that every term except the first is a sine or cosine. Integrating over a period will kill off everything but the first term, that is

\[ \int_{0}^{L} f(x) \, dx = \frac{a_0 L}{2} \]

To see how to get the remaining terms, we need to review some trig. Recall the identity

\[ \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \pm \sin \alpha \sin \beta \]

Adding and subtracting the two versions of this identity from each other yields

\begin{align*}
\cos(\alpha + \beta) + \cos(\alpha - \beta) &= 2 \cos \alpha \cos \beta \\
\cos(\alpha + \beta) - \cos(\alpha - \beta) &= 2 \sin \alpha \sin \beta
\end{align*}

Suppose now that \(\alpha = \frac{2m\pi x}{L}\) and \(\beta = \frac{2n\pi x}{L}\), where \(m\) and \(n\) are positive integers. Then

\begin{align*}
\cos \left( \frac{2m\pi x}{L} \right) \cos \left( \frac{2n\pi x}{L} \right) &= \frac{1}{2} \cos \left( \frac{2(m + n)\pi x}{L} \right) + \frac{1}{2} \cos \left( \frac{2(m - n)\pi x}{L} \right) \\
\sin \left( \frac{2m\pi x}{L} \right) \sin \left( \frac{2n\pi x}{L} \right) &= \frac{1}{2} \cos \left( \frac{2(m + n)\pi x}{L} \right) - \frac{1}{2} \cos \left( \frac{2(m - n)\pi x}{L} \right)
\end{align*}

What good is this? Since integrating cosine over a full period is zero, integrating the right-hand-side from 0 to \(L\) yields zero, that is

\[ \int_{0}^{L} \cos \left( \frac{2(m - n)\pi x}{L} \right) \, dx = 0 \]
so long as \( m \neq n \), and is \( L \) if \( m = n \). This shows that

\[
\int_0^L \cos \left( \frac{2m\pi x}{L} \right) \cos \left( \frac{2n\pi x}{L} \right) \, dx = 0
\]

\[
\int_0^L \sin \left( \frac{2m\pi x}{L} \right) \sin \left( \frac{2n\pi x}{L} \right) \, dx = 0
\]

unless \( m = n \neq 0 \), in which case it equals \( \frac{L}{2} \). The case \( m = n = 0 \) is slightly different; the first integral above yields \( L \), and the second is of course 0.

Furthermore, a similar argument starting from

\[\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta\]

leads to

\[
\sin \left( \frac{2m\pi x}{L} \right) \cos \left( \frac{2n\pi x}{L} \right) = \frac{1}{2} \sin \left( \frac{2(m+n)\pi x}{L} \right) + \frac{1}{2} \sin \left( \frac{2(m-n)\pi x}{L} \right)
\]

so that

\[
\int_0^L \sin \left( \frac{2m\pi x}{L} \right) \cos \left( \frac{2n\pi x}{L} \right) \, dx = 0
\]

for all integers \( m \) and \( n \).

We can now finally answer the original question: How does one obtain the other coefficients, simply multiply by the corresponding sine or cosine term and integrate. This will kill off all terms except one, which will be multiplied by \( \frac{L}{2} \). (This is the reason for the “extra” factor of 2 in the definition of \( a_0 \).)

At the end of the day, one obtains

\[
a_m = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{2m\pi x}{L} \right) \, dx
\]

\[
b_m = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{2m\pi x}{L} \right) \, dx
\]