WEEK 1 SUMMARY
Energy approach to equation of motion: i.e. find trajectory $x(t)$ if $U(x)$ is known

- Special case of $U(x)=(\frac{1}{2})kx^2$
  - found period $T$ (indep of $A$),
  - found $x(t) = A \cos(\omega t + \phi)$

- Found 3 other forms of $A \cos(\omega t + \phi)$
- Learned to apply initial conditions to determine $A$, $\phi$, and also the arbitrary parameters in other 3 forms
Energy approach to equation of motion:

- Harder case of \( U(\theta) = MgL_{cm}(1 - \cos \theta) \)
  - found \( \rightarrow \) HO for small theta
  - found \( T \) numerically
  - measured \( T \) for pendulum
- Learned to argue qualitatively about time to move certain distances, comparing \( T \) for diff \( U \)
- Did NOT learn equation of motion \( \theta(t) \)
Complex numbers:
• rectangular and polar form and Argand diag.
• complex conjugate
• Euler relation
\[ \exp(i\phi) = \cos\phi + i\sin\phi \]
• solving one complex equation is actually solving 2 simultaneous equations
WEEK 2 SUMMARY
Free, undamped oscillators

\[ m \ddot{x} = -kx \]

No friction

\[ \ddot{q} = -\frac{1}{LC} q \]

Common notation for all

\[ \ddot{\theta} \approx -\frac{g}{L} \theta \]

\[ \ddot{\psi} + \omega_0^2 \psi = 0 \]
Force approach to equation of motion of FREE, UNDAMPED HARMONIC OSCILLATOR:

\( i.e. \) find trajectory \( \theta(t) \) if \( F(\theta) \) is known

- Special case of \( F(\theta) = -\sin(\theta) \) \( \rightarrow \) small angle approx: \( F(\theta) = -\theta \) \( \rightarrow \) 2\(^{nd}\) order DE,

Found sinusoidal motion

\[
\theta(t) = Ce^{i\omega_0 t} + C^*e^{-i\omega_0 t}
\]

\[
\theta(t) = A \cos(\omega_0 t + \phi)
\]

Applied initial conditions as before.

\[
E = K + U = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} mgL\theta^2
\]
Free, damped oscillators

\[ m \ddot{x} = -kx - bx \]

\[
\begin{align*}
L \dot{I} + \frac{1}{C} q + RI &= 0 \\
\ddot{q} + \frac{R}{L} \dot{q} + \frac{1}{LC} q &= 0
\end{align*}
\]

Common notation for all

\[
\ddot{\theta} \approx -\frac{g}{L} \theta - b' \dot{\theta}
\]

\[
\ddot{\psi} + 2\beta \dot{\psi} + \omega_0^2 \psi = 0
\]
Force approach to equation of motion of FREE, DAMPED OSCILLATOR

- Add damping force to eqn of motion
- Found decaying sinusoid

\[ x(t) = Ce^{-(\beta + \omega_1)t} + C^*e^{-(\beta - \omega_1)t} \]

\[ = e^{-\beta t}\left[ Ce^{i\omega_1 t} + C^*e^{-i\omega_1 t} \right] \]

\[ x(t) = Ae^{-\beta t}\left[ \cos(\omega_1 t + \delta) \right] \]
FREE, DAMPED OSCILLATOR

- Damping time $\tau = 1/\beta$
- measures number of oscillations in decay time
- apply initial conditions, energy decay

$$Q = \pi \frac{\tau}{T} = \frac{\omega_0}{2\beta}$$

$$x(t) = Ae^{-\beta t} \left[ \cos(\omega_1 t + \delta) \right]$$
WEEK 3 SUMMARY
DRIVEN, DAMPED OSCILLATOR

\[ V_0 e^{i\omega t} - L\ddot{q} - \frac{q}{C} - R\dot{q} = 0 \]

\[ \ddot{q} + 2\beta \dot{q} + \omega_0^2 q = \frac{V_0}{L} e^{i\omega t} \]

\[ q(t) = \text{Re} \left[ |q_0| e^{i\phi_q} e^{i\omega t} \right] \]

\[ |q_0| = \frac{V_0/L}{\left[ (\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2 \right]^{1/2}}; \quad \tan\phi_q = \frac{-2\beta\omega}{\omega_0^2 - \omega^2} \]
\[ |q_0| = \frac{V_0/L}{\left[ (\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2 \right]^{1/2}} \]

\[ \phi_q = \arctan \left[ \frac{-2\beta\omega}{\omega_0^2 - \omega^2} \right] \]
\[ I(t) = \frac{dq(t)}{dt} = i\omega q(t) \]

Current Amplitude \( |I_0| \)

\[ |I_0| = \frac{\omega V_0 / L}{\left[ \left( \omega_0^2 - \omega^2 \right)^2 + 4\beta^2 \omega^2 \right]^{1/2}} \]

Driving Frequency \( \omega \rightarrow \)

Current Phase \( \phi_I \)

\[ \phi_I = \frac{\pi}{2} + \arctan \left[ \frac{-2\beta\omega}{\omega_0^2 - \omega^2} \right] \]
Admittance $Y(\omega) = \frac{I}{V_{app}}$

NOT time dependent, but IS freq dependent.

|Admittance Amplitude $|Y_0|$| \[|Y| = \frac{\omega/L}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2 \right]^{1/2}}\] |

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ο_0
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ο ----> (Driving Frequency)
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π/2
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0
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-π/2
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Addmittance Phase $\phi_I$
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\[\phi_I = \frac{\pi}{2} + \arctan\left[\frac{-2\beta\omega}{\omega_0^2 - \omega^2}\right]\]
DRIVEN, DAMPED OSCILLATOR

- can also rewrite diff eq in terms of I and solve directly (same result of course)

\[ V_0 e^{i\omega t} - L\ddot{q} - \frac{q}{C} - R\dot{q} = 0 \]

\[ \ddot{q} + 2\beta \dot{q} + \omega_0^2 q = \frac{V_0}{L} e^{i\omega t} \]

\[ \Rightarrow \ddot{q} + 2\beta \dot{q} + \omega_0^2 q = i\omega \frac{V_0}{L} e^{i\omega t} \]

\[ \dddot{I} + 2\beta \dot{I} + \frac{I}{LC} = i\omega \frac{V_0}{L} e^{i\omega t} \]
FOURIER SERIES – periodic functions are sums of sines and cosines of integer multiples of a fundamental frequency. These “basis functions are orthonormal

\[ f(t) = \sum_{n} a_n \cos n\omega t + b_n \sin n\omega t \]

\[ \frac{2}{T} \int_{0}^{T} \sin(p\omega t) \sin(q\omega t) \, dt = \delta_{pq} \]

\[ \frac{2}{T} \int_{0}^{T} \cos(p\omega t) \cos(q\omega t) \, dt = \delta_{pq} \]
ODD functions \( f(t) = -f(-t) \). Their Fourier representation must also be in terms of odd functions, namely sines.

Suppose we have an odd periodic function \( f(t) \) like our sawtooth wave and you have to find its Fourier series

\[
\sum_{n=1,2...} b_n \sin(n\omega t)
\]

Then the unknown coefficients can be evaluated this way

\[
b_n = \frac{2}{T} \int_{0}^{T} f(t) \sin(n\omega t) \, dt
\]

Integrate over the period of the fundamental

Here’ s the coefficient of the \( \sin(\omega_n t) \) term!
Plot it on your spectrum!

normalize properly the function the harmonic
The function

\[ f(t) = \sum_{n} \frac{2}{n\pi} \sin(n\omega_f t) \quad f(t) = A - \frac{2A}{T_f} t \quad 0 < t < T_f \]

Integrate over the period of the fundamental

\[ b_n = \frac{2}{T} \int_{0}^{T} f(t) \sin(n\omega t) \, dt \]

normalize properly

the function

the harmonic

Fundamental freq = \(2\pi/T\)

Coefficient of the \(\sin(\omega_n t)\) term!
DRIVING AN OSCILLATOR WITH A PERIODIC FORCING FUNCTION THAT IS NOT A PURE SINE

Important to know where the fundamental freq of the forcing function lies in relation to the oscillator max response freq!

Forcing function
DRIVING AN OSCILLATOR WITH A PERIODIC FORCING FUNCTION THAT IS NOT A PURE SINE

\[ V_{app} = V_0 e^{i\omega_f t} + 2V_0 e^{i2\omega_f t} \quad \text{(given)} \]

\[ \ddot{q} + 2\beta \dot{q} + \omega_0^2 q = V_0 e^{i\omega_f t} + 2V_0 e^{i2\omega_f t} \quad \text{(Kirchoff)} \]

\[ \Rightarrow q = q_\omega + q_{2\omega} \quad \text{(linear diff eq - superposition)} \]

\[ \Rightarrow I = I_{\omega_f} + I_{2\omega_f} \quad I = \dot{q} \]

\[ I = Y_{\omega_f} V_{app,\omega_f} + Y_{2\omega_f} V_{app,2\omega_f} \]

\[ I = \frac{\omega_f / L}{\left[ \left( \omega_0^2 - \omega_f^2 \right)^2 + 4\beta^2 \omega_f^2 \right]^{1/2}} e^{i \left( \frac{\pi}{2} + \arctan \left[ \frac{-2\beta \omega_f}{\omega_0^2 - \omega_f^2} \right] \right)} V_0 e^{i\omega_f t} \]

\[ + \frac{(2\omega_f)/L}{\left[ \left( \omega_0^2 - (2\omega)^2 \right)^2 + 4\beta^2 (2\omega)^2 \right]^{1/2}} e^{i \left( \frac{\pi}{2} + \arctan \left[ \frac{-2\beta 2\omega_f}{\omega_0^2 - (2\omega)^2} \right] \right)} 2V_0 e^{i(2\omega_f)t} \]
Observe what (LRC) black box does to an impulse function.

This was harmonic response expt - you know what black box (LRC) does to a single freq.

Are these connected by FT?? They’d better be - you find out!