PH424/525: 1-Dimensional Waves — Winter 2009

Homework #3 — Assigned Friday 02/13/09

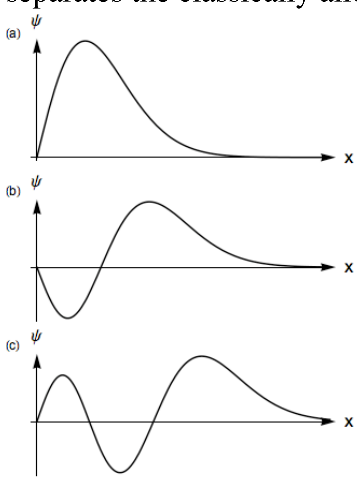
Q 1, 2 – due Wednesday 02/18/09

Q 3, 4,5 - Due Friday 02/20/09

1. Qualitative discussion of energy eigenfunctions

(a) The wavefunctions below are eigenstates of the Hamiltonian operator corresponding to a particular potential whose exact form is not important. Order the functions according the value of the corresponding eigenvalue, from lowest to highest, and explain why you made your choice.

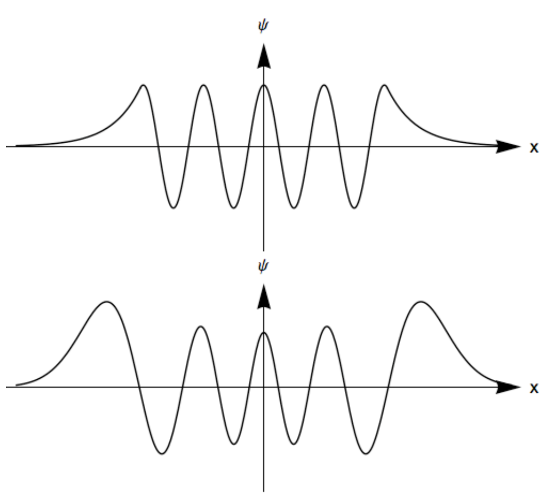
(b) Copy the graphs onto your answer sheet, and indicate on each graph the value of *x* that separates the classically allowed region from the classically forbidden region.



(c) Below are two eigenfunctions of *different* Hamiltonian operators. Consider the Hamiltonian operators corresponding to the 5 potential wells drawn below the functions.

For each function, decide to which Hamiltonian operators the function definitely does *not* belong, and say why. Then say why it could belong to the remaining one(s).

Eigenfunctions:



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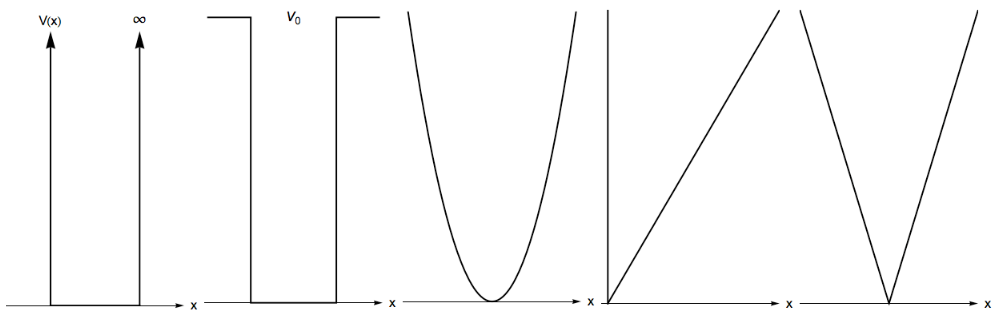
Potential energy functions: Please note that the *x*-scales for the potentials and the functions do *not* correspond to those of the wave functions. The potentials are just sketches to prompt you. They correspond to, from left to right, the infinite square well (ISW), the finite square well (FSW), the quadratic or harmonic oscillator potential (HO), and a 1-sided linear potential well (1LW), and a 2-sided linear well (2LW).



2. Measurement probabilities

A particle in an infinite square well potential of width L has a wave function at $t = 0$ given by

$$\psi(x,0) = \frac{1}{\sqrt{3}}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) - \frac{1}{\sqrt{3}}\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right) + \frac{i}{\sqrt{3}}\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right).$$

(i) Is the wave function normalized? If not, normalize it.
(ii) What are the possible outcomes of a measurement of the energy, and with what probability would they occur?
(iii) What is the average value of the energy?
(iv) Write down an expression for the wave function at some later time, t.
(v) At time $t = \frac{\hbar}{E_1}$, what are the possible outcomes of a measurement of the energy, and with what probability would they occur? (E_1 is the ground state energy.)

3. Expectation values

Calculate $\langle \hat{p} \rangle$, $\langle \hat{p}^2 \rangle$, $\langle x \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{H} \rangle$, $\Delta p \Delta x$ for an electron infinite square well potential of width L. (Potential is zero in the region $0<x<L$, infinite elsewhere.)
(i) in the first excited state
(ii) in an equal superposition of states n=3 and n=4
Note that $(\Delta p)^2 \equiv \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$ and similarly for Δx. This is what we mean by "uncertainty in momentum or uncertainty in position".

PARADIGMS IN PHYSICS

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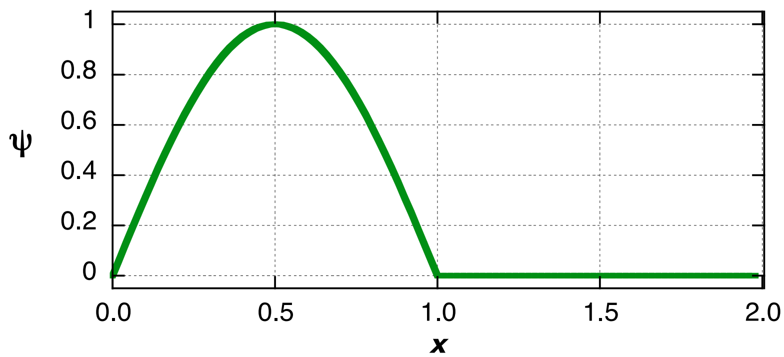
4. A problem about superposition and measurement:

(a) What is the lowest energy allowed for a particle of mass m that exists in an infinite square well potential of width $2L$? What would be the wave function describing the particle if it were in the lowest eigenstate?

(b) Suppose such a particle is *not* in an energy eigenstate of the well, but rather in some other state described the piecewise function $\psi(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) & 0 < x < L \\ 0 & L < x < 2L \end{cases}$.

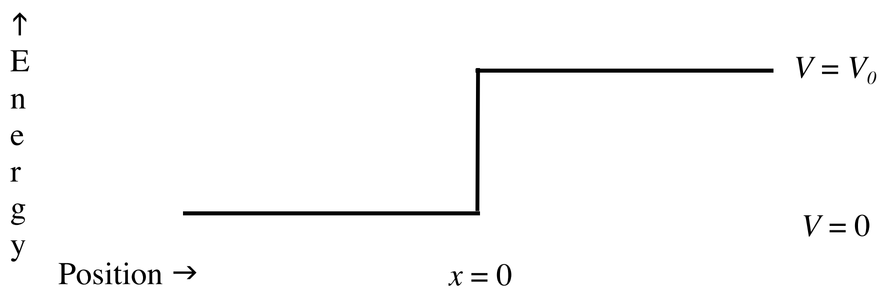
What is the probability that a measurement of the energy would yield the lowest allowed energy of the well?



5. Reflection of a quantum mechanical particle from a potential step

As you work through this problem, think carefully about the classical problems we discussed where a wave in a rope or cable is incident on a different rope or cable. At the end, you'll be asked to comment on the differences and similarities.

Let a particle of mass m and energy E be incident from the left on a barrier of height V_0. Consider the case $E > V_0$.



(A) Calculate the probability of reflection and transmission of this particle. Follow these steps …

(i) Set up a wave incident from the left, a wave reflected to the left, and one transmitted to the right, so that the total wave function is: $\psi(x,t) = \begin{cases} Ae^{-i(\omega t - k_1 x)} + Be^{-i(\omega t + k_1 x)} & x \le 0 \\ Te^{-i(\omega t - k_2 x)} & x \ge 0 \end{cases}$

Notice the use of the $-i\omega t$. We NEED the $-i\omega t$ to satisfy the time-dependent Schrödinger equation, but the *relative* sign of ω and t still tells the direction of travel: different signs for a wave propagating in the direction of increasing x (to the right by convention) and the same sign for a wave propagating in the direction of decreasing x (to the left by convention).

The frequency must be the same for all three terms – why?

Verify that these are indeed solutions to the Schrödinger equation for this problem, and in the process you will need to define the k-values in terms of parameters given in the problem (m, h, E and V_0).

(ii) Consider boundary conditions and establish the relationships among the coefficients.

(iii) We will now calculate something like our old friends the reflection and transmission coefficients, but it is <u>not useful</u> to calculate simply the ratios of the coefficients (the wave function amplitude is inherently not measurable). Instead we calculate something related to the intensity or probability – it is actually a particle current or particle flux

The flux is defined as $j = \frac{\hbar}{2im}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$

Note that this particle flux is proportional to the square of the wave function, and this is the measurable quantity (see Liboff section 7.5; Griffiths QM sort of discusses this in the discussion leading to Eqn 1.25).

Find j_{inc}, j_{ref} and j_{tran}, and calculate reflection and transmission coefficients

$r = \frac{j_{ref}}{j_{inc}}$ $\qquad\qquad$ $t = \frac{j_{tran}}{j_{inc}}$

(iv) Interpret your result for the case $V_0 = 0$.

(v) Interpret your result in the case where the particle's energy is much larger than the height of the barrier,

(vi) How do the cases $V_0 > 0$ and $V_0 < 0$ relate to your study of waves in ropes or cables?

(vii) How does the behavior of this quantum mechanical particle differ from that of a classical particle (one that obeys Newton's laws)? In other words, if a classical particle of energy E were to suddenly encounter a region where its potential energy changed, but not its total energy, what would happen?

6. (Optional) For those of you looking for a challenge. There are not many energy eigenvalue problems that are possible to solve analytically. But there are some.
   • Find the energy and the wavefunction (there is only one state) of a particle with energy E<0 in the delta function well : $V(x) = -V_0\delta(x)$. For this you will have to note a modification of a rule: The wave function derivative CAN be discontinuous if the potential energy is infinite at the point where the discontinuity occurs. (Remember the point mass on the rope?). This can be considered as the limit of the finite well problem we discussed in class, where the well becomes very deep and very narrow.