• In this assignment, it is highly desirable to use Maple to plot functions to gain insight into their shape and behavior, and to evaluate how well series approximate functions. It is also OK to use Maple to evaluate integrals, but you should not use the built-in Fourier coefficients functions, since this won't help you understand the process.

PRACTICE

1. Kronecker delta worksheet.
2. All of the integrals in the worksheet we did in class, both graphically and analytically.

REQUIRED

1. Prove analytically that $\frac{2}{T}\int_0^T \sin(n\omega t)\sin(m\omega t)\,dt = \delta_{m,n}$. Here $\omega = 2\pi/T$, n and m are integers and $\delta_{m,n}$ (the "Kronecker delta") is the function that is 1 if $m = n$ and 0 if $m \neq n$. (Do not choose specific values of m and n to show examples, prove in general for ANY m and n.) Hint: $\sin(n\omega t) = \frac{e^{in\omega t} - e^{-in\omega t}}{2i}$ will be useful to evaluate the integrals. You can use Maple to check your results, but it's important to do this analytically to aid your understanding of how Fourier series work.

   Use the Euler substitutions in the integral:

$$\frac{2}{T}\int_0^T \sin(n\omega t)\sin(m\omega t)\,dt = \frac{2}{T}\int_0^T \left(\frac{e^{in\omega t} - e^{-in\omega t}}{2i}\right)\left(\frac{e^{im\omega t} - e^{-im\omega t}}{2i}\right)dt$$

$$= \frac{2}{-4T}\int_0^T \left[e^{i(n+m)\omega t} + e^{-i(n+m)\omega t} - \left(e^{i(n-m)\omega t} + e^{-i(n-m)\omega t}\right)\right]dt$$

The integrals are easy, but we have to be careful when we deal with the m=n case, because there is a zero in the denominator.
First deal with the case $m \neq n$.

$$\frac{2}{T}\int_0^T \sin(n\omega t)\sin(m\omega t)\,dt = \frac{2}{-4T}\left[\left(\frac{e^{i(n+m)\omega t} - e^{-i(n+m)\omega t}}{i\omega(n+m)}\right) - \left(\frac{e^{i(n-m)\omega t} - e^{-i(n-m)\omega t}}{i\omega(n-m)}\right)\right]_0^T$$

$$= -\frac{2}{T}\left[\left(\frac{\sin\left[(n+m)\omega t\right]}{\omega(n+m)}\right) - \left(\frac{\sin\left[(n-m)\omega t\right]}{\omega(n-m)}\right)\right]_0^T$$

This follows because the integral of a sin function over a whole period is zero, and since n and m are integers, so too are n+m and n-m (and non-zero), thus the time T contains an integral number of periods of .

The case $m = n$ is special.
There are several ways to look at it. The easiest way is to avoid the zero denominator by setting m-n BEFORE you do the integral:

$$\frac{2}{T}\int_0^T \sin(n\omega t)\sin(m\omega t)\,dt = \frac{2}{T}\int_0^T \sin^2(m\omega t)\,dt$$
$$= \frac{2}{T}\int_0^T \left[\frac{1}{2}+\frac{1}{2}\cos 2m\omega t\right]dt$$
$$= \frac{2}{T}\left[\frac{T}{\ }+0\right] = 1$$

OR

$$\frac{2}{T}\int_0^T \sin(n\omega t)\sin(m\omega t)\,dt = \frac{2}{T}\int_0^T \left(\frac{e^{in\omega t}-e^{-in\omega t}}{2i}\right)\left(\frac{e^{im\omega t}-e^{-im\omega t}}{2i}\right)dt$$
$$= \frac{2}{-4T}\int_0^T \left[e^{i(2m)\omega t}+e^{-i(2m)\omega t}-\left(e^{i0\omega t}+e^{-i0\omega t}\right)\right]dt$$
$$= \ \ 2 \ \int^T [2 \cos 2m\omega t \ \ 2]\,dt = \ 2 \ \int^T 2\,dt = 1$$

If you do the integral first, before setting $m=n$, then you must be careful how you evaluate limits. Note that the integral with $n+m$ in the denominator evaluates to zero for the same reason as above, but the other term does not evaluate to zero! Let's look:

$$-\frac{2}{\ }\left[-\left(\frac{\sin\left[(n-m)\omega t\right]}{\ }\right)\right]^T = \frac{2\sin\left[(n-m)\omega T\right]}{\ } - \frac{2\sin\left[(n-m)\omega 0\right]}{\ }$$

Now we need to look at limits as $m \to n$. The numerator on the 2nd term right hand term is zero always because $t=0$. Thus as we take the limit $m \to n$, the term remains zero. The 1st term on the right goes to 1 as $m \to n$ as you can verify by L'Hopital's rule. We get the same result.

2.
Find by integration, the projection of the function onto (i) , (ii) , (iii) . Argue conceptually why this result is expected or not.

The integrals are straightforward – just do them. You'll find (i) 0 (ii) 0 (iii)1
This is expected because a trig identity allows you to rearrange the given function as $f(t) = 2\sin\omega t\cos\omega t = \sin 2\omega t$. Thus the projection onto itself must be 1 and onto all other independent functions must be zero.