The following exercises are due Friday, November 16.

1. Calculate $G(x_1, x_2)$ to second order using the $\frac{\lambda}{3!} : \varphi^3(x) :$ interaction.

2. Find the Fourier transform

$$\tilde{G}(p, p') = \int d^4 x \ e^{-ipx} \int d^4 y \ e^{ip'y} G(x, y)$$

Verify the form used in class

$$\tilde{G}(p, p') = (2\pi)^4 \delta^{(4)}(p-p')(-i\lambda)^2 i\Delta_F(p) \frac{d^4 q}{(2\pi)^4} i\Delta_F(q) i\Delta_F(p-q) i\Delta_F(p')$$

3. Calculate the S-matrix element for the process $k \to p_1 + p_2$ to lowest order in $\varphi^3$ theory. Since this process does not conserve momentum and energy, you must regard it as a virtual process. Work in momentum space directly using the Feynman rules. (This part is completely trivial. It’s just a prelude to the next question.)

4. Calculate the same process to third order. There are three one-particle reducible diagrams, which we already worked these out in class. You can ignore them because they just affect the propagator. The interesting diagram looks like a triangle. You will need the following integral formula

$$\frac{1}{abc} = \int_0^1 dz_1 \ dz_2 \ dz_3 \ \delta(1 - z_1 - z_2 - z_3) \ \frac{2}{[z_1 a + z_2 b + z_3 c]^3}$$
as well as the formula given in class

\[ \int \frac{d^d q}{(2\pi)^d} \frac{1}{(C^2 - q^2 - i\epsilon)^n} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left( \frac{1}{C^2} \right)^{n-d/2} \]

You are welcome to use Maple for the \( z \) integrals, but don’t do anything really heroic to finish the integration. The last integral may not be doable. At least I haven’t been able to do it. If you succeed I will be boundlessly impressed. What is the degree of divergence (or convergence) of the diagram? How does this affect the renormalization of the coupling constant?
Problem Set #2

1. \( A(\mathbf{x}_1, \mathbf{x}_2) = \int d^4x \int d^4y \; \text{LO} \left[ \mathcal{L}_\text{int} \left( \mathcal{L}_\text{int} \left( \mathcal{L}_\text{int} \left( \mathcal{L}_\text{int} \right) \right) \right) \right] \left( -i \gamma \right)^2 \left( -i \gamma \right)^2 \frac{1}{i \hbar} \left\langle \phi \left( \mathcal{L}_\text{int} \right) \phi \left( \mathcal{L}_\text{int} \right) \phi \left( \mathcal{L}_\text{int} \right) \phi \left( \mathcal{L}_\text{int} \right) \phi \left( \mathcal{L}_\text{int} \right) \right\rangle \)

Only fully contracted terms survive. We contract \( x_1 \) with either \( x_2 \) or \( y_1 \), contract two \( x_1 \)'s with two \( y_1 \)'s, and contract \( x_2 \) with the remaining \( x_2 \) or \( y \). There are \( 3! \times 3! \times 3! \) ways of doing this. We are left with

\[ O \left( A \right) = \frac{(-i \gamma)^2}{2} \int d^4x \int d^4y \; i \mathcal{D}(\mathbf{x}_1 - \mathbf{y}) i \mathcal{D}(\mathbf{y} - \mathbf{x}_2) i \mathcal{D}(\mathbf{x}_2 - \mathbf{x}_1) \]

\( \mathcal{D}_\text{int} \)

2. We can represent \( \mathcal{D}(\mathbf{x}) = \int \frac{d^4k}{(2\pi)^4} e^{i \mathbf{k} \cdot \mathbf{x}} \mathcal{D}_\text{int} \)

\[ \mathcal{D}_\text{int} = 1/(\mathbf{k}^2 - m^2 + i\epsilon) \]

Consequently, the \( \int d^4x \int d^4y \) integrals produce

\[ \int d^4x \int d^4y \; e^{i \mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{y})} e^{i \mathbf{k} \cdot (\mathbf{y} - \mathbf{x}_2)} e^{i \mathbf{k} \cdot (\mathbf{x}_2 - \mathbf{x}_1)} \]

\[ = (2\pi)^8 \delta^{(4)}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \delta^{(4)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) e \]

Now when we take the Fourier transform
we have the additional integrals
\[ \int d^4x_1 \int d^4x_2 e^{-i p x_1 - i p' x_2 + i \kappa_1 x_1 + i \kappa_2 x_2} \]
\[ = (2\pi)^8 \delta^{(4)}(p - \kappa_1) \delta^{(4)}(p' + \kappa_2) \]
So
\[ D(p, p') = \frac{\Lambda^2}{2} \int \frac{d^4\kappa_1}{(2\pi)^4} \delta^{(4)}(\kappa_1) \int \frac{d^4\kappa_2}{(2\pi)^4} \delta^{(4)}(\kappa_2) \]
\[ \times \right \frac{d^4\kappa_3}{(2\pi)^4} \delta^{(4)}(\kappa_3) \int \frac{d^4\kappa_4}{(2\pi)^4} \delta^{(4)}(\kappa_4) \]
\[ \times (2\pi)^6 \delta^{(14)}(p - \kappa_1) \delta^{(14)}(p' + \kappa_2) \delta^{(14)}(\kappa_2 + \kappa_3 + \kappa_4) \]
The \( \delta \)-functions require that \( \kappa_1 = -\kappa_2 = \rho = \rho' \).
so do the integrals over \( d\kappa_1 \) and \( d\kappa_3 \),
\[ D(p, p') = \frac{\Lambda^2}{2} \int \frac{d^4\kappa_2}{(2\pi)^4} \int d^4\kappa_3 \delta^{(14)}(\kappa_2 + \kappa_3 + \rho) \delta^{(14)}(p' + \kappa_2 + \kappa_3) \]
\[ \delta^{(14)}(p - \kappa_1) \delta^{(14)}(\kappa_1) \delta^{(14)}(\kappa_2) \delta^{(14)}(\kappa_3) \]
\[ = \frac{\Lambda^2 (2\pi)^4 \delta^{(14)}(p - p')}{{2}} \int \frac{d^4\kappa_2}{(2\pi)^4} \delta^{(14)}(p - \kappa_1) \delta^{(14)}(\kappa_1) \delta^{(14)}(\kappa_2) \delta^{(14)}(\kappa_3) \delta^{(14)}(p' + \kappa_2 + \kappa_3) \]
\[ = \frac{\Lambda^2 (2\pi)^4 \delta^{(14)}(p - p')}{{2}} \int \frac{d^4\kappa_2}{(2\pi)^4} \delta^{(14)}(p - \kappa_1) \delta^{(14)}(\kappa_1) \delta^{(14)}(\kappa_2) \delta^{(14)}(\kappa_3) \delta^{(14)}(p' + \kappa_2 + \kappa_3) \]
Problem Set # 2

Problem 3.

Feynman rules give

\[ S = - i \frac{(2\pi)^4}{3} S^{123} (k_1 - p_1 - p_2) \times \frac{i}{k^2 - m^2} \frac{i}{p_1^2 - m^2} \frac{i}{p_2^2 - m^2} \]

The only question concerns the combinatorial factors.

Think about calculating

\[ G(x_1, x_2, x_3) = \int dx_1 dx_2 dx_3 \sqrt{\text{det}(g)} \left[ \epsilon_{\mu_2}(x_1) \epsilon_{\mu_1}(x_2) \epsilon_{\mu_3}(x_3) \left( \frac{-i\gamma^\mu}{3!} \right) \delta(x_1 - x_2) \right] \]

We continue as follows:

\[ \overbrace{x_1, x_2, x_3}^{3} \overbrace{x, x, x}^{3} \overbrace{x, x}^{2} \]

So there are 3! terms canceling the 3! in the denominator. The formula for S is right as it stands.
Problem 4.

Now look at

\[
\begin{align*}
\langle 0 | T \left[ (e_1(x_1) e_2(x_2) e_3(x_3) : e^{(x)}_1 : e^{(x)}_2 : e^{(y)}_3 : e^{(y)}_4 : \right] & 10 \rangle \\
\text{Suppose we contract } x_1 \text{ with one of the } x's, \\
x_2 \text{ with one of the } y's, x_3 \text{ with the } y's. \text{ There are } 3^3 \text{ ways of doing this. This leaves}
\end{align*}
\]

\[
\begin{array}{c}
\text{ } \, x \, \, x \, + \, y \, + \, y \\
\text{connect } 1 \, x \, \text{ to any } y \, \text{ or } y \rightarrow 8 \text{ choices} \\
\text{remaining } x \, \text{ to remaining } y \, \text{ or } z \rightarrow 2 \text{ choices} \\
\text{two points } \rightarrow 1 \text{ choice}
\end{array}
\]

Finally there are 6 ways we could have chosen the first connection. In all,

\[
27 \times 16 \times 6 = 2592 \quad \text{possibilities}
\]

Since we are working to 3rd order, the S matrix contains the factors

\[
\frac{(-i)^3}{3!} \left( \frac{2}{3!} \right)^3 \times 2592
\]

\[
= 2^2 7^3
\]
F's rules then give

\[ 2i\lambda^3 \left( -\pi^4 \delta^{14} (\kappa_1 - p_1 - p_2) \right) \]

\[ \times (i)^3 \Delta_F(p_1) \Delta_F(p_2) \Delta_F(\kappa_1) \]

\[ \times \int \frac{d^4\kappa}{(2\pi)^4} \left( \frac{(i)^3}{\kappa^2 - m^2} \frac{(p_1 + \kappa)^2 - m^2}{(p_2 - \kappa)^2 - m^2} \right) \]
The real issue here is the integral

\[ X \left( \mathbf{k}, \mathbf{p}_2 \right) = \int \frac{d^4 \mathbf{k}}{(2\pi)^4} \frac{1}{\mathbf{k}^2 - m^2} \frac{1}{(\mathbf{p}_1 + \mathbf{k})^2 - m^2} \frac{1}{(\mathbf{p}_2 - \mathbf{k})^2 - m^2} \]

The denominator can be simplified using

\[ \frac{1}{abc} = \int_0^1 dz_1 \, dz_2 \, dz_3 \, \frac{2 \pi (1 - z_1 - z_2 - z_3)}{[g_1 a + g_2 b + g_3 c]^3} \]

\[ g_1 a + g_2 b + g_3 c = D = (\mathbf{k}^2 - m^2) g_1 \]

\[ + \left( p_1^2 + 2 \mathbf{k} \cdot \mathbf{p}_1 + \mathbf{k}^2 - m^2 \right) g_2 \]

\[ + \left( p_2^2 - 2 \mathbf{k} \cdot \mathbf{p}_2 + \mathbf{k}^2 - m^2 \right) g_3 \]

\[ = -m^2 (g_1 + g_2 + g_3) + \mathbf{k}^2 (g_1 + g_2 + g_3) \]

\[ + 2 \mathbf{k} \cdot (p_1 g_2 - p_2 g_3) + p_1^2 g_2 + p_2^2 g_3 \]

After we have done the \( z \) integrals, \( g_1 + g_2 + g_3 = 1 \)

Let's say we have already integrated over \( \int \, dz_1 \)

So \( g_2 + g_3 \) remain and are independent.

\[ D = \mathbf{k}^2 + 2 \mathbf{k} \cdot \mathbf{p} + \Delta \]

Where \( \Phi = p_1 g_2 - p_2 g_3 \) and \( \Delta = p_1^2 g_2 - p_2^2 g_3 - m^2 \)

Now complete the square

\[ D = (\mathbf{k} + \mathbf{\Phi})^2 + \Delta \]

Shifting the integration contour gives
\[ D = \pi^2 - \rho^2 + \Delta \]

Now use the integral formula
\[
\int \frac{d^n q}{(2\pi)^n} \frac{1}{(c^2 - q^2 - i\varepsilon)^n} = i^{n-d/2} \frac{\Gamma \left( \frac{n-d}{2} \right)}{\Gamma(n)} \left( \frac{1}{c^2} \right)^{n-d/2}
\]

In our case \( d = 4 \), \( n = 3 \), \( n - d/2 = 1 \)

\[
\int \frac{d^4 \tau}{(2\pi)^4} \frac{1}{(p^2 - \Delta - i\varepsilon)^3} = i \frac{1}{(4\pi)^2} \cdot \frac{1}{\rho^2 - \Delta}
\]

\[
\overline{X}(p, \rho) = i \frac{1}{2 \cdot (4\pi)^2} \int_0^1 d\tau_2 d\tau_3 \frac{1}{(p^2 - \Delta - i\varepsilon)}
\]

Look at the denominator
\[
\Delta = p^2 - \Delta = -p_1^2 \rho_2 + p_2^2 \rho_3 + \rho^2 + p_1^2 \rho_0 - 2p_1p_2 \rho_2 \rho_3 + p_2^2 \rho_3
\]

\[
= p_1^2 \rho_2 (\rho_2 - 1) + p_2^2 \rho_3 (\rho_3 + 1)
\]

\[
+ \rho^2 - 2p_1p_2 \rho_2 \rho_3
\]

\[
= \rho^2 + (p_1 \rho_2 - p_2 \rho_3)^2 + (p_2 \rho_3 - p_1 \rho_2)
\]

So
\[
\overline{X}(p, \rho) = i \frac{1}{2 \cdot (4\pi)^2} \int_0^1 d\tau_2 d\tau_3 \frac{1}{\rho^2 + (p_1 \rho_2 - p_2 \rho_3)^2 + (p_2 \rho_3 - p_1 \rho_2)}
\]

Neither Maple nor I know how to do the two integrations.