These problems are due Friday, February 22.

1. An ideal circular parallel plate capacitor of radius $a$ and plate separation $d \ll a$ is connected to a current source by axial leads. The current in the wire is $I(t) = I_0 \cos \omega t$. (Jackson problem 6.14)

   (a) Calculate the electric and magnetic fields between the plates to second order in powers of the frequency (or wave number), neglecting the effects of fringing fields.

   (b) Calculate the volume integrals $w_e$ and $w_m$ that enter in the definition of the reactance $X$ to second order in $\omega$. As we showed in class

   $$X = \frac{4\omega}{|I_i|^2} \int_V (w_m - w_e) d^3x$$

   $$w_e = \frac{1}{4}(\mathbf{E} \cdot \mathbf{D}^*) \quad w_m = \frac{1}{4}(\mathbf{B} \cdot \mathbf{H}^*)$$

   Show that

   $$\int w_e d^3x = \frac{|I_i|^2 d}{\omega^2 a^2} \quad \int w_m d^3x = \frac{|I_i|^2 d}{8c^2} \left(1 + \frac{\omega^2 a^2}{12c^2}\right)$$

   Here $I_i = -i\omega Q$, where $Q$ is the total charge on one plate.

2. Consider the Dirac expression

   $$\mathbf{A}(\mathbf{R}) = -g \int_{-\infty}^{0} d\mathbf{l}' \times \nabla(1/|\mathbf{r} - \mathbf{r}'|)$$

   for the vector potential of a magnetic monopole and its associated string as discussed in class. Suppose as we did that the monopole is located at the origin and the string along the negative $z$ axis. (Jackson 6.18)
(a) Verify my claim that

\[ A = \frac{g\hat{\phi}}{r} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \]

(b) Verify that \( B = \nabla \times A \) is the Coulomb-like field of a point charge, except perhaps at \( \theta = \pi \).

(c) With the \( B \) determined in part b, evaluate the total magnetic flux passing through the circular loop of radius \( R \sin \theta \) centered on the \( z \) axis. Consider \( \theta < \pi/2 \) and \( \theta > \pi/2 \) separately, but always calculate the upward flux.

(d) From \( \oint A \cdot dl \) around the loop, determine the total magnetic flux through the loop. Compare the result with that found in part c. Show that they are equal for \( 0 < \theta < \pi/2 \), but have a constant difference from \( \pi/2 < \theta < \pi \). Interpret this difference.
Use Maxwell's eqn. For harmonic fields \(6.130\)
Between the plates \(c = \mu = 1\) and \(J = 0\)
\(B = \vec{B}_0\) \(\vec{B}_e\), and \(E = \vec{E}_0\) \(\vec{E}_e\).

\[
\begin{align*}
(1) & \quad \nabla \times \vec{E} = \frac{i}{\omega} \vec{B} \\
(2) & \quad \nabla \times \vec{B} = -\frac{i}{\omega} \vec{E} \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \vec{E}_e}{\partial \rho} = -\frac{i}{\omega} \vec{B}_0 \\
\frac{\partial \vec{B}_e}{\partial \rho} = -\frac{i}{\omega} \vec{E}_0
\end{align*}
\]

(a) An electric field due to \(\vec{E}_0\) on the plates creates a magnetic field through (2). This \(\vec{B}_0\) in turn produces a new electric field through (1).
The new \(\vec{E}\) differs from the old \(\vec{E}\) by a factor \(\omega^2 / c^2\). Repeating this procedure gives an infinite series in powers of \(\omega^2 / c^2\).

\[
I(t) = I_0 \cos \omega t = \text{Re} \left[ I_0 e^{-i\omega t} \right]
\]

\[
\frac{dQ(t)}{dt} = I(t) \quad Q(t) = \text{Re} \left[ \frac{iI_0 e^{-i\omega t}}{\omega} \right]
\]

So \(Q(t) = \text{Re} \left[ Q_0 e^{i\omega t} \right]\) where \(Q_0 = \frac{iI_0}{\omega}\)

From here on \(\vec{E}, \vec{O}, \vec{E}, \vec{B}\) etc. refer to the time-independent quantities. (Complex fields)

\[
E_2^{(0)} = \frac{4\pi \Phi}{\pi a^2} = \frac{4\pi}{c^2} = \frac{4\pi I_0}{\omega a^2}
\]

From (2) \(\oint_{\partial S} \vec{B} \cdot d\vec{l} = -\frac{i}{\omega} \int \vec{E} \cdot \vec{n} \, d\sigma\)

Taking the surface \(S\) to be a circle of radius \(\rho\):

\[
2\pi \rho B_{\rho}^{(2)} = \left( -\frac{i}{\omega} \right) \left( \frac{4\pi I_0}{\omega a^2} \right) \pi \rho^2
\]

\[
B_{\rho}^{(2)} = \frac{2\rho}{a^2 c} I_0
\]
From (1) \[ E_z^{(2)} = -i\frac{\omega}{c} \int B_\ell d\rho \]

\[ = -i\frac{\omega}{c} \left( \frac{2\eta_0}{a^2 c} \right) \int \rho d\rho = -i\frac{\omega \rho^2 \eta_0}{a^2 c} = -\frac{\omega^2 \rho^2}{4c^2} E_z^{(0)} \]

Again from (2) \[ \oint B_\ell \cdot d\mathbf{e} = -i\frac{\omega}{c} \int E \cdot \mathbf{n} \, d\alpha \]

\[ 2\pi \rho B_\ell^{(3)} = \left( -i\frac{\omega}{c} \right) \int \rho \left( -i\frac{\omega \rho^2 \eta_0}{a^2 c} \right) 2\pi \rho' d\rho' \]

\[ B_\ell^{(3)} = -\frac{\omega^2 \rho^3 \eta_0}{4a^2 c^3} = -\frac{\omega^2 \rho^2}{8c^2} B_\ell^{(1)} \]

So \[ E_z \cdot E_z^{(0)} + E_z^{(1)} = \frac{4i\eta_0}{\omega a^2} \left( 1 - \frac{\omega^2 \rho^2}{4c^2} \right) \]

\[ B_\ell \cdot B_\ell^{(1)} + B_\ell^{(3)} = \frac{2\rho \eta_0}{a^2 c} \left( 1 - \frac{\omega^2 \rho^2}{8c^2} \right) \]

(b) This is not the whole story, however, not even to order $\omega^2$. To all orders of $\omega$ it must be true that $\nabla \cdot \mathbf{E} = 4\pi \rho$, or $E = 4\pi \rho$. So having corrected $E$ to second order we must go back and correct $\mathbf{Q}$.

\[ \mathbf{Q} = \frac{1}{4\pi} \mathbf{E} = \frac{i\eta_0}{\pi \omega a^2} \left( 1 - \frac{\omega^2 \rho^2}{4c^2} \right) \]

\[ Q = \int_0^\alpha \int_0^\alpha 2\pi \rho d\rho = \frac{2i\eta_0}{\omega a^2} \int_0^\alpha \left( 1 - \frac{\omega^2 \rho^2}{4c^2} \right) \rho d\rho \]

\[ = \frac{i\eta_0}{\omega} \left[ 1 - \frac{\omega^2 a^2}{8c^2} \right] = R_0 \left[ 1 - \frac{\omega^2 a^2}{8c^2} \right] \]

Define \[ I_\ell = -2\omega Q = \eta_0 \left[ 1 - \frac{\omega^2 a^2}{8c^2} \right] \]
In the equations for $E$-$B$ we replace

$$J_0 \rightarrow \frac{I_i}{1 - \frac{w^2 a^2}{8 c^2}}.$$  

The energies are then given by

$$W_e = \frac{1}{16 \pi} \int_{V} \left| E \right|^2 dV \propto \frac{\int_{V} \sqrt{d} \, dV}{\pi w^2 a^4} \int_0^\alpha \left[ 1 - \frac{w^2 c^2 + w^4 c^4}{2 c^4} \right] 2 \pi r^2 dr \propto \frac{d}{w^2 a^2} \left[ 1 - \frac{w^2 a^2}{4 c^2} \right] \propto \frac{d}{w^2 a^2} \frac{I_i}{I_i^2} \frac{l_i}{l_i^2} \frac{l_i^2}{l_i^2}.$$  

$$W_m = \frac{1}{16 \pi} \int_{V} \left| B \right|^2 dV \propto \frac{\int_{V} \sqrt{d} \, dV}{4 \pi a^4 c^2} \int_0^\alpha \left[ 1 - \frac{w^2 c^2 + w^4 c^4}{4 c^2} \right] 2 \pi r^2 dr \propto \frac{d}{8 c^2} \left[ 1 - \frac{w^2 a^2}{6 c^2} \right] \propto \frac{I_i}{8 c^2} \left[ 1 + \frac{w^2 a^2}{12 c^2} \right].$$

while we have systematically discarded all terms in $w^4$ and higher powers.

\[ \text{(c)} \]

From eq. (6.140)

$$X = \frac{4W}{I_i^2} (W_m - W_e)$$

$$= \frac{wd}{2c^2} - \frac{4d}{wa^2} + O(w^3)$$

$$\approx \frac{wL}{wc} - \frac{1}{wc}$$

so \[ L \approx \frac{d}{2c^2} \quad C = \frac{a^2}{4d} \]

$$W_{ns} = \frac{1}{VLC} = \frac{2\sqrt{2}}{c/a} = 2.828 \frac{c}{a}$$

The first root of $J_0$ is $2.405$. 

2. \(a)\)

In cylindrical coord

\[ R^2 = (y-y')^2 + \rho^2 \]

\[ \frac{\partial}{\partial \rho} \left( \frac{1}{R} \right) = \frac{\partial}{\partial \rho} \left[ (y-y')^2 + \rho^2 \right]^{-1/2} \]

\[ = - \rho \left[ \rho^2 + (y-y')^2 \right]^{-3/2} \]

\[ A = - q \int_{-\infty}^{\infty} d\rho \times \nabla \left( \frac{1}{R} \right) \]

\[ \hat{\rho} \times \hat{\rho} = \hat{\theta} \]

\[ \hat{\theta} \times \hat{\rho} = -\hat{\rho} \]

\[ = q \int_{-\infty}^{\infty} d\rho' \left[ \rho^2 + (y-y')^2 \right]^{-3/2} \]

\[ = q \int_{-\infty}^{\infty} \frac{d\rho}{\rho^2 + (y-y')^2} \left[ \frac{y-y'}{\rho^2 + (y-y')^2} \right]_{-\infty}^{0} = \frac{q \epsilon (1-\cos \theta)}{r \sin \theta} \]

b) New go to polar coordinates

\[ \nabla \times A = \hat{r} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \ A_{\theta} \right) \]

\[ + \hat{\theta} \left[ - \frac{1}{r} \frac{\partial}{\partial r} \left( r A_{r} \right) \right] \]

\[ = \hat{r} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{q (1-\cos \theta)}{r} \right) = \hat{r} \frac{q}{r^2} \]
(a) To simplify the integration, assume that the "surface" of the loop is the section of a sphere with radius R and opening angle $\Theta$

For $0 < \Theta < \pi / 2$

$$\Phi = \int B \cdot n \, ds = \frac{\mu_0}{R^2} \int \hat{R} \cdot d\Omega$$

$$= 2\pi \frac{\mu_0}{R^2} \cos \Theta \Bigg|_{\cos \Theta} = 2\pi \Phi (1 - \cos \Theta)$$

The situation for $\frac{\pi}{2} < \Theta < \pi$ is more complicated.

First, the solenoid carries flux upward through the loop. The total flux radiated by the monopole is

$$\Phi_{\text{total}} = \frac{\Phi}{r^2} \times 4\pi R^2 = 4\pi \Phi$$

so the string carries this flux through the loop. Second, the flux from the monopole passes downward through the loop, so it must be subtracted from the total.

Finally, the opening angle of the solid angle cone is $\pi - \Theta$. Put all this together.

$$\Phi = 4\pi \Phi - 2\pi \Phi \left[ 1 - \cos (\pi - \Theta) \right]$$

$$= 2\pi \Phi (1 - \cos \Theta)$$
1) \[
\oint A \cdot dl = \frac{q}{RSin\theta} (1 - \cos\theta) \times 2\pi RSin\theta
= 2\pi q (1 - \cos\theta)
\]

The last question is puzzling. If we had neglected to add the contribution from the solenoid, and if we had neglected to change the sign of the flux through the loop, then our answer would be wrong by the constant $2\pi q$. 