Problem Set #2

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These problems are due Friday, Oct. 19.

1. The generating function expansion for the Legendre polynomials is

\[ V = (1 - 2hz + h^2)^{-1/2} = \sum_{l=0}^{\infty} h^l P_l(z) \]

where \( 0 < h < 1 \) and \( z = \cos(\theta) \). By expanding the left side of this equation in powers of \( h \), prove that

\[ P_l(z) = \max_r \sum_{r=0}^{r_{\text{max}}} (-1)^r (2l - 2r)! \frac{2^l}{2^r! (l - r)! (l - 2r)!} z^{l - 2r} \]

where \( r_{\text{max}} = l/2 \) or \( (l - 1)/2 \), whichever is an integer.

2. Prove that the polynomials \( P_l(z) \) given by the formula above satisfy Legendre’s equation

\[ \frac{d}{dz} \left[ (1 - z^2) \frac{dP_l}{dz} \right] + l(l + 1) P_l = 0 \]

3. Given the generating function above

   a) Verify the formula

   \[ (1 - 2hz + h^2) \frac{\partial V}{\partial h} = (z - h)V \]

   and show by comparing powers of \( h \) on both sides of the equation that

   \[ (l + 1)P_{l+1} - (2l + 1)z P_l + lP_{l-1} = 0 \]
(b) Similarly, starting with the equation

\[ \frac{h}{\partial h} \frac{\partial V}{\partial h} = (z - h) \frac{\partial V}{\partial z} \]

derive a recursion relation for the first derivatives of \( P_l \).
The generating function expansion is

\[ V = (1 - 2h y + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{h^n}{n!} \Phi(n) \]

Denote \( y = 2h y - h^2 \) and expand \( V \) in powers of \( y \).

\[ V = (1 - y)^{-\frac{1}{2}} \quad \frac{dV}{dy} = \frac{1}{2} (1 - y)^{-3/2} \]

\[ \frac{d^n}{d^ny} V = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{(2n-1)}{2} \left(1 - \frac{y}{2}ight)^{-\frac{1}{2}(2n+1)} \]

so \( \frac{d^n}{d^ny} V(0) \bigg|_{y=0} = \frac{(2n-1)!!}{(2^n n!)^2} = \frac{(2n)!}{2^{2n} n!} \)

where we have used

\( (2n-1)!! = \frac{(2n)!}{(2^n n)!} \)

so \( V = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n! n!} \frac{y^n}{n!} \)

Now expand \( y \) in the binomial expansion.

\[ y^n = \sum_{i=0}^{n} \frac{n!}{i! (n-i)!} (-1)^{n-i} \left(\frac{y}{2h}\right)^i \frac{n!}{i! (n-i)!} \]

Inserting in the expression for \( V \) gives
\[ V = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n! n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (-1)^{j} (8g)^{n-j} h^{j} \]

Let \( n+j = l \) we would like to print out all the terms of the series that give us any particular \( l \). Moreover \( j \leq n \). The trick is to replace the sums over \( n \) and \( j \) with sums over \( l = n+j \) and a new index \( \tau = j - l-n \). The maximum value \( \tau \) can take is either \( l/2 \) or \( (l-1)/2 \) whichever is an integer.

\[ V = \sum_{l=0}^{\infty} \sum_{\tau=0}^{\tau_{\text{max}}} (-1)^{\tau} \frac{(2l-2\tau)!}{2^{l-\tau} \tau! (l-\tau)! (l-2\tau)!} h^{l} \]

By definition \( P_e(g) \) is the coefficient of \( h^l \). This proves the theorem.
Further explanation of limits:

This sum \[ \sum_{n=0}^{\infty} \sum_{i=0}^{l} \] with \( l = n + \hat{i} \) contains the terms

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
2 & 0 & 2 & 0 \\
2 & 1 & 3 & 0 \\
2 & 2 & 4 & 0 \\
3 & 0 & 3 & 0 \\
3 & 1 & 4 & 0 \\
3 & 2 & 5 & 0 \\
3 & 3 & 6 & 0 \\
4 & 0 & 4 & 0 \\
4 & 1 & 5 & 0 \\
4 & 2 & 6 & 0 \\
4 & 3 & 7 & 0 \\
4 & 4 & 8 & 0 \\
\end{array}
\]

\( \ldots \)

which is the same as \[ \sum_{n=0}^{\infty} \sum_{i=0}^{\text{rmax}} \]

where \( \hat{i} = 0 \) if \( \hat{i} \leq \text{rmax} \)

which can be reorganized as follows...
\[ P_e = \sum_{n=0}^{v_{\text{max}}} \frac{(-1)^n (2l-2n)!}{2^n n! (l-n)! (l-2n)!} \quad \beta' = \sum c(l,r) \beta^r \]

\[ P' = \sum c(l-2r) \beta^{2r} \]

\[ P'' = \sum c(l-2r)(l-2r-1) \beta^{2r+1} \]

so \( (1-\beta^2) P' = 2\beta P + l(l+1) P = 0 \)

becomes \( \sum c(l-2r)(l-2r-1) \beta^{2r} \]

\[ + \sum c \left[ \frac{l(l+1)}{2} - 2(l-2r) - (l-2r)(l-2r-1) \right] \beta^{2r} = 0 \]

In the first sum replace \( r' = r+1 \) so that \( \beta^{2r} \rightarrow \beta^{2r'} \). Thus

\[ \sum_{r'=1}^{r'} c(l, r'-1) (l-2r'+2)(l-2r'+1) \beta^{2r'} \]

\[ + \sum_{r=0}^{l} c(l, r) 2r(l+2l-2r) \beta^{2r} = 0 \]

Now look at \( c(l, r'-1) = \frac{(-1)^{r'-1} (2l-2r'+2)!}{2^l (r'-1)! (l-r'+1)! (l-2r'+2)!} \)

\( c(l, r'-1)(l-2r'+2)(l-2r'+1) = \frac{(-1)^{r'-1} (2l-2r'+2)!}{2^l (r'-1)! (l-r'+1)! (l-2r')!} \)

\[ = \frac{(-1)^{r'} (2l-2r')! (2l-2r'+1)(2l-2r'+2) r'}{2^l r'! (l-r')! (l-r'+1) (l-2r')!} \]

\[ = -c(l, r') 2 (2l-2r'+1) r' \]
Note that this term = 0 when \( r' = 0 \). Now drop primes in first term and set lower limit = 0

\[
\sum_{r' = 1} c (l, r' - 1) (l - 2r' + 2) (l - 2r' + 1) \frac{Z}{l - 2r'}
\]

\[
\Rightarrow - \sum_{r' = 0} c (l, r) 2r (l + 2l - 2r) \frac{Z}{l - 2r}
\]

So the two terms cancel and the d.e. is satisfied.
2a) \[ V = (1 - 2h^2 + h^4)^{-1/2} \]

\[ V^{-2} \frac{\partial V}{\partial h} = (y - h) V^3 \times V^{-2} \]

\[ (1 - 2h^2 + h^4) \frac{\partial V}{\partial h} = (y - h) V \]

So \[ (1 - 2h^2 + h^4) \sum_e h^{e-1} P_e = (y - h) \sum_e h^e P_e \]

\[ 0 = \sum_e h^{e+1} (e+1) P_e - \sum_e (2e + g) h^e P_e \]

\[ + \sum_e e h^{e-1} P_e = 0 \]

First term: set \( e+1 = e' \)

Third term: set \( e-1 = e'' \)

\[ 0 = \sum_{e'=0}^\infty h^{e'} e' P_{e'-1} - \sum_{e=0}^\infty g (2e+1) h^e P_e + \sum_{e''=0}^\infty (e''+1) h^{e''} P_{e''+1} \]

so \( e P_{e-1} - g (2e+1) P_e + (e+1) P_{e+1} = 0 \)

2b) \[ h \frac{\partial V}{\partial h} = h (y - h) V^3 \]

\[ (y - h) \frac{\partial V}{\partial y} = (y - h) h V^3 \]

\[ h \frac{\partial V}{\partial h} = h \sum_e e h^{e-1} P_e = (y - h) \frac{\partial V}{\partial y} = (y - h) \sum_e h^e P_e \]

\[ 0 = \sum_e h^{e+1} P_e + \sum_e (e P_e - g P_e) h^e = 0 \]
\[ 0 = \sum_{l' = 1}^{\infty} h^{l'} \dot{p}_{l'1} + \sum_{l = 1}^{\infty} h^{l} \left( lp_{l} - \frac{a}{2} \dot{p}_{l} \right) = 0 \]

So
\[ \dot{p}_{l-1} + lp_{l} - \frac{a}{2} \dot{p}_{l} = 0 \]

or
\[ \dot{p}_{l} + (l+1)p_{l+1} - \frac{a}{2} \dot{p}_{l+1} = 0 \]