1. Consider a boundary value problem defined inside a conducting sphere of radius $R$. The sphere is divided into two hemispheres with opposite potentials $V$ and $-V$. Derive a formula for the potential everywhere inside the sphere in terms of the constant $V$. Do this in two different ways:

(a) Assume that the “northern” and “southern” hemispheres are at opposite potentials so that $\phi(r = R) = V$ for $0 < \theta < \pi/2$ and $\phi(r = R) = -V$ for $\pi/2 < \theta < \pi$ (35 points)

(b) Assume that the “eastern” and “western” hemispheres are at opposite potentials so that $\phi(r = R) = V$ for $0 < \phi < \pi$ and $\phi(r = R) = -V$ for $\pi < \phi < 2\pi$ (35 points)

In both cases you will have integrals over $\theta$ that are not easy to do. **Do not try to evaluate these integrals.** In both cases the integrals will vanish for some values of $m$ and $l$. Be sure to specify these terms and explain why they vanish.

2. I have described below three typical electrostatics problems. I would like you to explain how you would solve them. In each case describe how you would set up the coordinate system, what functions you would use, and how you would use them. Please note: **you are not to solve the problems, just explain how you would solve them.** (10 points each)

(a) Find the potential between two concentric spheres. The potential is known (and not zero) on the two spherical surfaces, and there is no charge between them.

(b) The potential is specified on the surface of an infinitely long cylinder, the potential has no $z$-dependence, and there is no charge outside the cylinder. Find the potential outside of the cylinder.
(c) The charge density is specified on the surface of an infinite plane. This charge vanishes i.e. $\sigma \to 0$ at distances far from the origin. There is no charge elsewhere. Find the potential above the surface.

Here are some possibly useful formulas.

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2}$$

$$\phi(r, \theta, \varphi) = \sum_{lm} \left[ A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta, \varphi)$$

$$\phi(r, \theta) = \sum_l \left[ A_l r^l + B_l r^{-(l+1)} \right] P_l(\theta)$$

$$\phi(r, \theta, \varphi) = k \int \frac{\rho(r')}{|r - r'|} dV'$$

$$\int Y_{lm}^*(\theta, \varphi) Y_{l'm'}(\theta, \varphi) d\Omega = \delta_{ll'} \delta_{mm'}$$

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\varphi}$$

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{m} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

$$\phi(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{\nu=1}^{\infty} (a_\nu \rho^{\nu} + b_\nu \rho^{-\nu}) [A_\nu \cos(\nu \varphi) + B_\nu \sin(\nu \varphi)]$$

$$\int_0^{2\pi} d\varphi \sin(n\varphi) \sin(m\varphi) = \int_0^{2\pi} d\varphi \cos(n\varphi) \cos(m\varphi) = \pi \delta_{mn}$$

$$\int_0^{2\pi} d\varphi \cos(n\varphi) \sin(m\varphi) = 0$$
1. as usual \( c = \sum_{l=0}^{\infty} A_{2l} \rho^{2l} Y_{2l} (\theta, \phi) \) at \( r = R \)

\[ V(\theta, \phi) = \sum_{l,m} A_{lm} R^l Y_{lm} \]

(a) No \( \theta \) dependence - assume \( V(\theta) = \sum_{l} A_{l} R^{l} P_{l} \)

\[ \int V(\theta) P_{l}(\cos \theta) \sin \theta d\theta = A_{l} R^{l} \frac{\pi}{2l+1} \]

\[ A_{l} = \frac{2l+1}{2 \pi R^{l}} \int_{-1}^{1} V(\theta) P_{l}(\theta) d\theta \]

\[ = \frac{2l+1}{2 \pi R^{l}} \left[ \int_{0}^{1} P_{l}(\theta) d\theta - \int_{-1}^{0} P_{l}(\theta) d\theta \right] \]

\[ = \frac{2l+1}{2 \pi R^{l}} \int_{0}^{1} \left[ P_{l}(\theta) - P_{l}(-\theta) \right] d\theta \]

\[ = 0 \quad \text{if } l \text{ is even} \]

\[ = \frac{2l+1}{2 \pi R^{l}} \int_{0}^{1} P_{l}(\theta) d\theta \quad l \text{ odd} \]

\( c (\psi, \theta, \phi) = \sum_{l, m} (2l+1) \left( \frac{\psi}{R} \right)^{2l} A_{lm} \int_{0}^{1} P_{l}(\theta) d\theta \)
(b) Now we have $V$ dependence

$$V(e) = \sum_{m=1}^{\infty} A_m R^e Y_m$$

$$\int_{0}^{\pi} V(e) Y_m^*(e,e) \, e^{-i \omega t} \, d\omega = A_m R^e$$

$$\int_{0}^{\pi} \int_{-1}^{1} \frac{2e+1}{(e-m)!} P^m_e(z) \left\{ \int_{-\pi}^{\pi} e^{i \omega t} \, d\omega - \int_{0}^{\pi} e^{-i \omega t} \, d\omega \right\} \, dz \, dc$$

$$= \int_{0}^{\pi} \int_{-1}^{1} \frac{2e+1}{(e-m)!} P^m_e(z) \left( -\frac{4iz}{m} \right) \, dz \, dc$$

$$= 0 \quad \text{if } m \text{ even}$$

Don't try to integrate this -

Call it $V_{em}$

$$A_{em} = R^{-e} V_{em}$$

$$\Phi(v,e,e) = \sum_{e=1}^{\infty} \sum_{m \text{ odd}} \left( \frac{v}{R} \right)^e Y_m \, V_{em}$$
\( \phi = \sum_{n=1}^{\infty} \left[ A_n r^n + B_n r^{-(n+1)} \right] Y_{n} \ (r \leq a) \)

We need both \( A_n \) and \( B_n \) terms.

Use orthogonality of the \( Y_n \)'s

\( \) and solve simultaneously for \( A_n \) and \( B_n \).

\( \phi = A_0 \log r + \sum_{n=1}^{\infty} \rho^{-n} \left[ A_n \cos n \theta + B_n \sin n \theta \right] \)

Find the \( A_0 \) term from

\[ \int_0^{2\pi} \phi (r=a) \, d\theta = A_0 \frac{2\pi}{n} \ln a \]

Since \[ \int_0^{2\pi} \cos n \theta \, d\theta = \int_0^{2\pi} \sin n \theta \, d\theta = 0 \]

Find the other terms using orthogonality of sines and cosines.

\( \) (c) Since the charge is given

\[ \phi = \kappa \int \frac{p(r') \, dV'}{(r^2 - r'^2)^{3/2}} \]

Analytically, you might try

\[ \frac{1}{r^2 - r'^2} = \sum_{\kappa=1}^{\infty} \frac{4\pi}{2\kappa+1} \gamma_{\kappa}^* (1/r) \gamma_{\kappa} (6/r') \left( \frac{r'}{r} \right)^{2\kappa+1} \]

But this will only converge for large \( r \), close to the center you would have to do this numerically.