1. Please answer the following five qualitative questions with a brief explanation. (10 points each)

(a) Explain how much and what kind of information is required to pose an electrostatics problem with a unique solution.

(b) The integral
\[ \phi(r) = \int \frac{\rho(r')}{|r-r'|} dr' \]
gives the potential \( \phi(r) \) everywhere in space. Why do we need all the other stuff about potential expansions and Green’s functions?

(c) What is a Dirichlet Green’s function? What conditions must it satisfy? How do you use it?

(d) Answer the same questions with regard to the Neumann Green’s function.

(e) The potential due to a unit point charge at the origin in three dimensions is
\[ \phi(r) = \frac{1}{4\pi\epsilon_0 r}. \]
It would seem natural that the potential due to a line charge, i.e. no \( z \)-dependence, would be proportional to \( 1/\rho \), but instead it is
\[ \phi(\rho) = -\frac{\lambda}{2\pi\epsilon_0} \ln(\rho). \]

Why?
2. The two dimensional region shown below is bounded by the sides $\varphi = 0$, $\varphi = \beta$, and $\rho = R$. A known potential $V(\varphi)$ is given on the “surface” $\rho = R$, and the other two sides are held at zero potential. Find the potential inside. Note: You don’t have to find the Green’s function (although that’s one way to do the problem), you just have to find the potential. In order to insure success, I would like you to follow the steps below.

(a) Use the separation of variables technique to find all those solutions to Laplace’s equation (in two dimensions) that vanish at $\varphi = 0$ and $\varphi = \beta$. Which of these solutions are relevant to this problem?
(b) Use this information to find a general formula for the potential.

3. (a) Find the Dirichlet Green function for the inside of the two-dimensional region $0 \leq x \leq a$ and $0 \leq y \leq a$. There are several ways to do this, but the eigenvalue technique would be easy. (25 points)
(b) If your Green’s function is correct it must satisfy the basic equation $\nabla^2 G(x, y; x', y') = -4\pi \delta(x-x')\delta(y-y')$, or to put it another way,
\[
\nabla \int f(x', y')G(x, y; x', y')dx'dy' = -4\pi f(x, y)
\]
where $f(x, y)$ is any function that vanishes on the sides of the square. Show that your Green’s function has this property. (25 points)

4. A hollow sphere of radius $R$ carries a surface charge on its inside surface. The “northern” hemisphere carries a uniform surface charge density $\sigma$. The “southern” hemisphere carries an equal and opposite surface charge density $-\sigma$. There is no other charge in the problem. Find the potential inside the sphere. (50 points)
Here are some (possibly) useful formulas.

\[ \nabla^2 \phi(\rho) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} \]

\[ \phi(r) = \sum_{l=0}^{\infty} [a_l r^l + b_l r^{-(l+1)}] P_l(\cos(\theta)) \]

\[ \int_{-1}^{1} P_l(z) P_{l'}(z) \, dz = \frac{2 \delta_{ll'}}{2l+1} \]

\[ (2l + 1) P_l(z) = \frac{d}{dz} P_{l+1}(z) - \frac{d}{dz} P_{l-1}(z) \]

\[ \phi(r) = \int_V \rho(r') G(r, r') \, dr' + \frac{1}{4\pi} \int_S \left[ G(r, r') \frac{\partial \phi(r')}{\partial n'} - \phi(r') \frac{\partial G(r, r')}{\partial n'} \right] \, da' \]

\[ G(r, r') = -4\pi \sum_n \frac{\phi_n(r) \phi_n(r')}{\lambda_n} \]

\[ \phi(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_n \rho^n \left[ c_n \cos n\varphi + s_n \sin n\varphi \right] + \sum_n \rho^{-n} \left[ C_n \cos n\varphi + S_n \sin n\varphi \right] \]

\[ \int_S \mathbf{E} \cdot \mathbf{n} \, dS = Q/\epsilon_0 \]