Summary of canonical transformations

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- Lagrangian mechanics

\[ L(q, \dot{q}) = T - V \quad \text{no velocity-dependent forces} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \]

- Hamiltonian dynamics – We always assume that energy is conserved and there is no explicit time dependence in the Hamiltonian.

\[ H(q, p) = \sum_{k=1}^{n} p_k \dot{q}_k - L \]

\[ p_k \equiv \frac{\partial L}{\partial \dot{q}_k} \quad \dot{q}_k = \frac{\partial H}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \]

- Canonical transformations. The idea is to find a new set of dynamic variables in terms of which the problem becomes trivial. In general the Hamiltonian is transformed according to the following scheme.

\[ K(Q_1, \ldots, Q_n, P_1, \ldots, P_n) = H(q_1(Q, P), \ldots, q_n(Q, P), p_1(Q, P), \ldots, p_n(Q, P)) + \frac{\partial F}{\partial t} \]

We always use \( F = F_2(q, P) \), that is to say a function of the old coordinates and the new momenta. In this case

\[ p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i} \]

These will be coupled equations. They can be combined in such a way as to express the new variables as functions of the old or vice versa. Unfortunately, the \( F_2 \) generating function gets called \( S \) or \( W \) depending on context.

There are several strategies for transforming the Hamiltonian.
Hamilton’s principal function. The idea is to make \( K = 0 \) so that \( \dot{Q} = \dot{P} = 0 \). Then \( P \equiv \alpha \) is a constant of motion, and \( Q \equiv \beta \) depends on the initial conditions. The Hamiltonian-Jacobi equation in one dimension is then

\[
H \left( q, \frac{\partial S}{\partial q} \right) + \frac{\partial S}{\partial t} = 0
\]

In more than one dimension we assume separability, i.e.

\[
S = \sum_i W_i(q_i) - \alpha t
\]

finally

\[
S = S(q_1 \cdots q_n; \alpha_1 \cdots \alpha_n; t)
\]

where all the \( \alpha_i \)'s are independent constants of motion.

Hamilton’s characteristic function. Now \( K = \alpha \). Naturally \( \alpha \) is the total energy and thus a constant of motion. In one dimension

\[
K(P) = H \left( q, \frac{\partial S}{\partial q} \right) = E = P \equiv \alpha = \text{constant}
\]

\[
S = S(q, P) \quad p = \frac{\partial S}{\partial q} \quad Q = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \alpha} \equiv \beta
\]

\[
\dot{\beta} = \frac{\partial K}{\partial \alpha} = 1 \quad \beta = t - t_0
\]

In several dimensions

\[
S = \sum_k W_k(q_k, \alpha_1, \cdots, \alpha_n) \equiv W
\]

\[
K = H \left( q, \frac{\partial W}{\partial q} \right) = E \equiv \alpha_1
\]

Separate variables and identify the other constant \( P_k \equiv \alpha_k \).

\[
\dot{Q}_k = \frac{\partial K}{\partial P_k} = \frac{\partial K}{\partial \alpha_k} \equiv \dot{\beta}_k = \delta_k \quad \beta_k = \frac{\partial W}{\partial \alpha_k}
\]

These are the equations of motion.

Action-angle variables. The philosophy is similar to that above except the canonical momentum is chosen to be the action rather than the total energy.

\[
K(I) = E = H \left( q, \frac{\partial S}{\partial q} \right)
\]
\[ I = \frac{1}{2\pi} \oint p \, dq \quad S = \int p \, dq \]

\[ \psi = \frac{\partial W}{\partial I} \quad p = \frac{\partial W}{\partial q} \]

\[ \dot{i} = -\frac{\partial}{\partial \psi} K(I) = 0 \quad \dot{\psi} = \frac{\partial}{\partial I} K(I) = \omega(I) \]

In several variables

\[ W(q_1, \ldots, q_n; I_1, \ldots, I_n) = \sum_j W_j(q_j; I_1, \ldots, I_n) \]

\[ I_i = \frac{1}{2\pi} \oint p_i \, dq_i \]

\[ \psi_i = \frac{\partial W}{\partial I_i} \quad \dot{\psi}_i = \frac{\partial K(I_1, \ldots, I_n)}{\partial I_i} = \omega_i \]

The \( \dot{\psi} \)'s are all constant, so the \( \psi \)'s are all linear functions of time as in the one-dimensional case.