Solution for week 7

PDF version of solutions

1. **Energy of a relativistic Fermi gas** There are a couple of ways you could go through this problem. One would be to just integrate to find the Fermi energy, and then to integrate to find the internal energy. It’s not bad done that way. The other way, which I’ll demonstrate, is to first solve for the density of states, and then use that to find the Fermi energy and $U$.

$$D(\varepsilon) = 2 \left( \frac{L}{2\pi} \right)^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\varepsilon(k) - \varepsilon) d^3k$$

$$= 2 \left( \frac{L}{2\pi} \right)^3 \int_{0}^{\infty} \delta(hck - \varepsilon) 4\pi k^2 dk$$

Note that the factors of two above are for the spin degeneracy. Now changing variables to an energy:

$$\varepsilon = hck$$

$$d\varepsilon = hcdk$$

And we get

$$D(\varepsilon) = 8\pi \left( \frac{L}{2\pi} \right)^3 \left( \frac{1}{hc} \right)^3 \int_{0}^{\infty} \delta(\varepsilon - \varepsilon) \varepsilon^2 d\varepsilon$$

$$= 8\pi \left( \frac{L}{2\pi hc} \right)^3 \varepsilon^2$$

$$= \frac{V}{\pi^2 h^3 c^3} \varepsilon^2$$

a) Solving for the Fermi energy comes down to solving for $N$.

$$N = \int_{0}^{\varepsilon_F} D(\varepsilon) d\varepsilon$$

$$= \frac{V}{\pi^2 h^3 c^3} \int_{0}^{\varepsilon_F} \varepsilon^2 d\varepsilon$$

$$= \frac{V}{\pi^2 h^3 c^3} \frac{1}{3} \varepsilon_F^3$$

$$\varepsilon_F = \left( \frac{N}{V} 3\pi^2 h^3 c^3 \right)^{\frac{1}{3}}$$

$$= (3\pi^2 n)^{\frac{1}{3}} hc$$

just as the problem says. The dimensions are energy because $n^{\frac{1}{3}}$ is inverse length, which when multiplied by $c$ gives inverse time. $h$ is energy times time, so we get an energy as we expect.

b) The internal energy at zero temperature (which is the total energy of the ground state) just requires us to just integrate the density of states time energy.

$$U = \int_{0}^{\varepsilon_F} D(\varepsilon) \varepsilon d\varepsilon$$

$$= \int_{0}^{\varepsilon_F} \frac{V}{\pi^2 h^3 c^3} \varepsilon^3 \varepsilon d\varepsilon$$

$$= \frac{V}{\pi^2 h^3 c^3} \frac{4}{3} \varepsilon_F^4$$

$$= \left( \frac{V}{\pi^2 h^3 c^3} \frac{4}{3} \varepsilon_F \right)^N$$

$$= 3^4 N \varepsilon_F$$

The trickiest step was looking back at our previous expression for $N$ to substitute.

2. **Pressure and entropy of a degenerate Fermi gas**

a) As we saw before (when working with the radiation pressure of a vacuum?) the pressure is given by the thermal average value of the derivative of the energy eigenvalues.
\[ p = - \left( \frac{\partial U}{\partial V} \right)_{S,N} \tag{17} \]
\[ = \sum_i P_i \left( - \frac{\partial E_i}{\partial V} \right) \tag{18} \]

The usual challenge here is that fixed temperature is not the same thing as fixed entropy. In this case, when \( T = 0 \), we know that the probabilities are all predetermined (via the Fermi-Dirac distribution), and we can just take a simple derivative of the energy we derived in class.

\[ U = \frac{3}{5} N \varepsilon_F \tag{19} \]
\[ = \frac{3}{5} N \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{\frac{2}{3}} \tag{20} \]

\[ p = - \left( \frac{\partial U}{\partial V} \right)_{S,N} \tag{21} \]
\[ = \frac{3}{5} N \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{\frac{2}{3}} \frac{1}{3} \frac{2}{V} \tag{22} \]
\[ = \frac{2}{5} N \frac{\hbar^2}{V} \varepsilon_F \tag{23} \]
\[ = \frac{1}{5} \frac{\hbar^2}{m} \frac{\pi^{\frac{3}{2}}}{3} \left( \frac{N}{V} \right)^{\frac{5}{3}} \tag{24} \]

This agrees with the expression given in the problem itself, so yay.

b) The entropy is a Fermi gas, when \( kT \ll \varepsilon_F \). We can start with the general form of entropy:

\[ S = -k \sum_i P_i \ln P_i \tag{25} \]

We will begin by first finding the entropy of a single orbital, and then adding up the entropy of all the orbitals. One orbital has only two microstates, occupied and unoccupied, which correspondingly have probabilities \( f(\varepsilon) \) and \( 1 - f(\varepsilon) \). Before we go any farther, it is worth simplifying the latter.

\[ 1 - f = 1 - \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \tag{26} \]
\[ = \frac{e^{\beta(\varepsilon - \mu)} + 1}{e^{\beta(\varepsilon - \mu)} + 1} - \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \tag{27} \]
\[ = \frac{e^{\beta(\varepsilon - \mu)} + 1}{e^{\beta(\varepsilon - \mu)} + 1} \tag{28} \]
\[ = \frac{1}{e^{-\beta(\varepsilon - \mu)} + 1} \tag{29} \]

The entropy corresponding to a single orbital, thus is

\[ S_{\text{orbital}} = -k(f \ln f + (1 - f) \ln(1 - f)) \tag{30} \]
\[ = -k \left( \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \ln \left( \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \right) + \frac{1}{e^{-\beta(\varepsilon - \mu)} + 1} \ln \left( \frac{1}{e^{-\beta(\varepsilon - \mu)} + 1} \right) \right) \tag{31} \]
\[ = \frac{k}{e^{\beta(\varepsilon - \mu)} + 1} \ln \left( e^{\beta(\varepsilon - \mu)} + 1 \right) \]
\[ + \frac{k}{e^{-\beta(\varepsilon - \mu)} + 1} \ln \left( e^{-\beta(\varepsilon - \mu)} + 1 \right) \tag{32} \]

This is inherently symmetric as we change the sign of \( \varepsilon - \mu \), which makes sense given what we know about the Fermi-Dirac distribution. It is less obvious in this form that the entropy does the right thing (which is to approach zero) when \( |\varepsilon - \mu| \gg kT \). We expect the entropy to go to zero in this case, and one term very obviously goes to zero, but the other requires a bit more thinking. A simple approach is to plot the entropy, as I do below, which demonstrates that the entropy does indeed vanish at energies far from the Fermi level.

Using this expression for the entropy of a single orbital, we can solve for the entropy of the whole gas. At the second step below we will make use of the fact that \( kT \ll \varepsilon_F \), which means that the
The entropy looks very much like a Dirac $\delta$-function that hasn’t been properly normalized.

$$S = \int_0^\infty D(\varepsilon) S_{\text{orbital}}(\varepsilon) d\varepsilon$$

$$= D(\varepsilon_F) \int_{-\infty}^\infty S_{\text{orbital}}(\varepsilon) d\varepsilon$$

$$= D(\varepsilon_F) \int_{-\infty}^\infty \left( \frac{k}{e^{\beta(\varepsilon - \mu)} + 1} \ln \left( e^{\beta(\varepsilon - \mu)} + 1 \right) + \frac{k}{e^{-\beta(\varepsilon - \mu)} + 1} \ln \left( e^{-\beta(\varepsilon - \mu)} + 1 \right) \right) d\varepsilon$$

This looks nasty, but we can make it dimensionless, and it’ll just be a number!

$$\xi = \beta(\varepsilon - \mu) \quad d\xi = \beta d\varepsilon$$

which gives us

$$S = D(\varepsilon_F) k^2 T \int_{-\infty}^{\infty} \left( \frac{\ln(e^\xi + 1)}{e^\xi + 1} + \frac{\ln(e^{-\xi} + 1)}{e^{-\xi} + 1} \right) d\xi$$

Now the last bit is just a number, which happens to be finite.

### 3. White dwarf

a) Showing that something is a given order of magnitude can be both tricky and confusing. The potential energy is exactly given by

$$U = -\frac{1}{2} \int d^3 r \int d^3 r' \frac{G \rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

as you learned in static fields, where $\rho(\vec{r})$ is the mass density. Unfortunately, we don’t know what the mass density is as a function of position, or how that function depends on the mass of the white dwarf.

An adequate if not satisfying set of reasoning is to say that the integral above must scale as $M^2$, since increasing $M$ will either increase $\rho$ or increase the volume over which the mass is spread. The denominator $|\vec{r} - \vec{r}'|$ is going to on average scale as the radius $R$. Thus it makes sense that the potential energy would be about $\sim -\frac{GM^2}{R}$. Another approach here would have been to use dimensional analysis to argue that the energy must be this. Alternatively, you could have assumed a uniform mass density, and then argued that the actual energy must be of a similar order of magnitude.

b) The kinetic energy of the electrons is the $U$ of the Fermi gas, which in class we showed to be

$$KE \sim N \varepsilon_F$$

$$\sim N \frac{\hbar^2}{2m} \left( \frac{N}{V} \right)^\frac{3}{2}$$

$$\sim \frac{\hbar^2 N^{5/3}}{mR^2}$$

where $m$ is the mass of the electron. Then we can reason that the number of electrons
is equal to the number of protons, and if the star is made of hydrogen the total mass of the star is equal to the total mass of its protons.

\[ N \approx \frac{M}{M_H} \quad (42) \]

\[ KE \sim \frac{\hbar^2 M^\frac{2}{3}}{m M_H^\frac{2}{3} R^2} \quad (43) \]

c) At this stage is is worth motivating the virial theorem from mechanics, which basically says that the magnitude of the average potential energy of a bound system (which is bound by a power law force) is about the same as the average of its kinetic energy. This makes sense in that the thing that is holding a bound state together is the potential energy, while the thing that is pulling it apart is the kinetic energy. If they aren’t in balance, then something weird must be going on. BTW, this virial theorem also applies quite well to the quantum mechanical hydrogen atom.

All right, so

\[ \frac{GM^2}{R} \sim \frac{\hbar^2 M^\frac{2}{3}}{m M_H^\frac{2}{3}} \quad (44) \]

\[ M^\frac{2}{3} R \sim \frac{\hbar^2}{m M_H^\frac{2}{3} G} \quad (46) \]

\[ R^3 \sim \left( \frac{\hbar^2}{m M_H^\frac{2}{3} G} \right)^3 \frac{1}{M} \quad (47) \]

\[ \rho = \frac{M}{\frac{4}{3} \pi R^3} \quad (48) \]

\[ \sim M^2 \left( \frac{\hbar^2}{m M_H^\frac{2}{3} G} \right)^3 \quad (49) \]

Plug in numbers.

c) All that changes is that our degenerate gas has mass \( M_N \approx M_H \), and our total mass is now \( M = NM_N \approx NM_H \).

\[ \frac{GM^2}{R} \sim \frac{\hbar^2 M^\frac{2}{3}}{M_H M_H^\frac{2}{3}} \quad (50) \]

\[ M^\frac{2}{3} R \sim \frac{\hbar^2}{M_H^\frac{2}{3} G} \quad (51) \]

\[ R \sim \frac{\hbar^2}{M^\frac{2}{3} M_H^\frac{2}{3} G} \quad (52) \]

Plug in numbers to find the neutron star radius in kilometers when its mass is one solar mass.

\[ R \sim \left( \frac{10^{-27} \text{g cm}^2}{s} \right)^2 \left( \frac{10^{-24} \text{g}}{s^2} \right)^2 \left( \frac{10^{-7} \text{cm}^3}{s^2} \right) \quad (53) \]

\[ \sim 10^6 \text{cm} \sim 10 \text{km} \quad (54) \]

That’s what I call a small-town star!

4. Fluctuations in the Fermi gas We are looking here at a single orbital, and asking what is the variance of the occupancy number.
\[
\langle (\Delta N)^2 \rangle = \langle (N - \langle N \rangle)^2 \rangle = \sum_i P_i (N_i - \langle N \rangle)^2
\] (55)
\[
\langle (\Delta N)^2 \rangle = P_1 (1 - f)^2 + P_0 (0 - f)^2 = f(1 + (2f - 2f) + (1 - f) f^2 = f - f^2 = \langle N \rangle (1 - \langle N \rangle) \] (56)
Now this single orbital has only two possible states: occupied and unoccupied! So we can write this down pretty quickly, using the probability of those two states, which are \( f \) and \( 1 - f \). We also note that \( \langle N \rangle = f \), so we’re going to have \( f \) all over the place.
\[
\langle (\Delta N)^2 \rangle = f (1 + f^2 - 2f) + (1 - f) f^2 = f - f^2 = \langle N \rangle (1 - \langle N \rangle) \] (57)
This tells us that there is no variation in occupancy when the occupancy reaches 0 or 1. In retrospect that is obvious. If there is definitely an electron there, then we aren’t uncertain about whether there is an electron there.

5. **Einstein condensation temperature** I’m going to be sloppier on this solution, because this is done in the textbook, and I’m still quite sick. The idea is to set the chemical potential to 0, which is its maximum value and integrate to find the number of atoms not in the ground state, \( N_E \), which is normally essentially equal to the number of atoms total.
\[
N_E = \int_0^\infty \mathcal{D}(\varepsilon) f(\varepsilon) d\varepsilon = \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{\xi}}{e^{\xi} - 1} d\xi \] (61)
\[
N_E = \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \int_0^\infty \varepsilon^{1/2} \frac{1}{e^{\varepsilon} - 1} d\varepsilon \] (62)
Naturally at this stage we will want to use a change of variables to take the physics out of the integral as we typically do.
\[
N_E = \frac{V}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{\xi}}{e^{\xi} - 1} d\xi \] (63)
Now we can simply solve for \( T_E \).
\[
T_E = \frac{1}{k_B} \frac{\hbar^2}{2M} \left( \frac{N}{V} \frac{4\pi^2}{\int_0^\infty \frac{\sqrt{\xi}}{e^{\xi} - 1} d\xi} \right)^{2/3} \] (64)